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THE MATHEMATICS TEACHER

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The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry*

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INTRODUCTION

THAT mathematics is in some way intimately bound up with logic has been recognized since the days of Euclid, who, without explicit formulation, utilized laws of logic in many of his indirect proofs. The important place that mathematics has occupied unchallenged for centuries in the curricula of secondary schools, colleges, and universities is undoubtedly due to an unconscious and unquestioning faith in the power of mathematics to train the mind in logical thinking in other branches of knowledge. With the rise of educational psychology in the beginning of the twentieth century, experimental evidence began to accumulate against the possibility of automatic transfer of training from one subject to another unless "identical elements" were present in both subjects.[1] Whereupon a contrary reaction set in. The formulation of the "law of specificity of training" was surely one of the major causes that brought about, in the last decade, the decline of the prestige of mathematics as the irreplaceable preparation for clear thinking and logical reasoning.

That the truth lies somewhere between the two extreme points of view has always been evident to the more critical students of the controversy. Judd and Orata have very effectively presented the point of view that, in general, transfer is possible but is not automatic, and that identical elements are not all easily discernible even to the bright pupil, and they become discernible only after careful analysis and correlation.

The real problem of transfer is a problem of so organizing training that it will carry over in the minds of students into other fields. There is a method of teaching a subject so that it will transfer, and there are other methods of teaching the subject

[1] For an excellent summary and critical evaluation of the relevant literature see: Betz, William, "The Transfer of Training with Particular Reference to Geometry," *Fifth Yearbook of the National Council of Teachers of Mathematics*, pp. 149–98. Bureau of Publications, Teachers College, Columbia University. 1930.

See also Orata, Pedro T., (1) *The Theory of Identical Elements*. Ohio State University, Columbus, Ohio. 1928. (2) "Transfer of Training and Educational Pseudo-Science," *The Mathematics Teacher*, 28: 265–89, May 1935. (3) "Transfer of Training and Reconstruction of Experience," *The Mathematics Teacher*, 30: 99–109, March 1937.

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so that the transfer will be very small. Mathematics as a subject cannot be described in my judgment as sure to transfer. All depends upon the way in which the subject is handled. . . . Transfer is a form of generalization, and training can be given so as to encourage generalization, or training can be of such a type as to hinder generalization.[2]

Accepting the point of view presented above, the author of the present study will attempt to show that it is possible to utilize geometry as a medium for making the pupils conscious of the existence of logical patterns of valid and invalid reasoning in mathematics as well as in the thinking of everyday life.

To be sure, many textbooks in geometry make some reference to the frequent falsity of the converse of a proposition and to the nature of indirect reasoning. Some books even mention the "opposite" of a proposition, and provide some exercises in its formation. But, unfortunately, these transformations do not appear with enough frequency throughout the course in geometry to leave a lasting impression on the student's habits of thinking.

[2] Judd, C. H., quoted in *The Reorganization of Mathematics in Secondary Education*, pp. 99–100. The Mathematical Association of America, 1923.

The objects of this study are as follows:

1. The traditional definitions of the converse are critically examined and another definition is proposed. The advantages of the latter are indicated.

2. A treatment similar to the above is given to the concept of the "opposite" of a proposition. The importance of the latter concept for everyday thinking is pointed out.

3. The revival of the notion of the contrapositive of a theorem in the teaching of geometry is urged. The exposition of the traditional definition of the contrapositive and of the Law of Contraposition is followed by an enumeration of their shortcomings. A wider definition of the concept is proposed and the proof of a more generalized law of contraposition is established. The importance of such an approach both for the mathematician and for the teacher of mathematics is demonstrated.

Chapter I

THE CONVERSE

The Terms: "Hypothesis," "Given," and "Data"

The word *hypothesis*, fortunately, is going out of use in mathematics textbooks for secondary pupils. It is, first, too long and too strange a word for a simple idea. Second, its meaning in the natural sciences and in general conversation is not quite the same as that in geometry, and this difference tends to confuse the beginner; third, its singular form tends to give the beginner the impression that there is one hypothesis, instead of a set of *hypotheses*, each one of which is of equal importance in contributing to the inevitability of the conclusion.

Unfortunately, the word *given* that replaced hypothesis suffers from the same defect of being singular, and, in addition, has no plural form at all. As a result writers and teachers have had to resort to the circumlocutions "one of the given conditions," "one of the conditions given in the hypothesis," or "one of the hypotheses." The writer would, therefore, like to urge the adoption of the word *data*, and the phrase, *one of the data of the theorem*, or simply *datum*. The term *data* which has the same meaning as *hypothesis* and *given* has the advantages not only of being plural in form, but also of being in common use in such phrases as "statistical data," "scientific data," and "accurate data."[1]

[1] Auerbach, M. and Walsh, C. B., *Plane Geometry*. J. B. Lippincott Co. 1920. On page 56 the word *data* is used in the sense recommended here.

Historical Survey of the Occurrence of the Term[2]

Euclid. Strange as it may seem, Euclid did not use the term converse, or its equivalent, and he never pointed out in his *Elements* that certain theorems were converses of others. One searches in vain for that familiar term either in the text of Todhunter's edition of Euclid or in the text of the more authoritative, and perhaps definitive version edited by Heath.[3]

Modern Textbooks. With the introduction of geometry into the secondary schools, the number of textbooks on Euclid for that level grew enormously. In this study 93 books were examined, and in each the term *converse*, or one of its equivalents, always appears. After stating the definition of the term, the author usually adds the statement that a converse is not necessarily true and that it therefore requires independent proof. The form of the statement ranges from a bald statement of fact without example or proof, found in the earlier textbooks, to a praiseworthy attempt at convincing the student of the invalidity of converses by means of well-chosen examples, found in most textbooks published in recent years.

The Uni-Converse Interpretation

The Definition of a Converse of a Proposition in Contemporary Textbooks

The definition of the converse of a theorem found in all but eight of the textbooks examined was to the effect that it is obtained by interchanging the hypothesis and the conclusion. The following typical definitions are taken from two standard textbooks:

1. If two propositions are so related that what is given in each is what is to be proved in the other, each proposition is called the converse of the other.[4]
2. A proposition is the converse of another when the hypothesis and the conclusion of the one are respectively the conclusion and the hypothesis of another.[5]

Although none of the books states it explicitly, the phrasing of the definition forces upon the reader the implication that every theorem has but one converse. An examination of the converses stated in those books bears out the above impression.

Such an interpretation of the term *converse* that ascribes to a theorem but one converse, will be referred to hereafter in this essay as the *uni-converse approach.*

Critique of the Uni-Converse Approach

The persistence of the traditional definition despite the recurrent convincing arguments against it may not render totally unnecessary the recital of the following criticisms, as well as of others that are mainly of a pedagogic nature.

Proclus. Fifteen hundred years ago Proclus (410–485 A.D.) pointed out the existence of two forms of conversion, *conversion par excellence* and *partial conversion.*

The leading form, the conversion *par excellence*, is the complete or simple conversion in which the hypothesis and the conclusion of a theorem change places exactly, the conclusion of the theorem being the hypothesis of the converse theorem, which again establishes, as its conclusion, the hypothesis of the original theorem. . . .

The other form of conversion, which we may call *partial*, is seen in cases where a theorem starts from two or more hypotheses combined into one enunciation and leads to a certain conclusion, after which the converse theorem takes this conclu-

[2] Since this essay is primarily critical in nature, no attempt is made to present an exhaustive historical study of each topic treated. Only so much of the background will be sketched in as to afford material for analysis and evaluation.

[3] Todhunter, I., *The Elements of Euclid*, with an Introduction by Sir Thomas L. Heath. Everyman's Library, No. 891. E. P. Dutton and Co. 1933.

Heath Thomas L., *The Thirteen Books of Euclid's Elements*, with Introduction and Commentary. Cambridge University Press. 3 Vols., Second edition. 1926.

[4] Wentworth, G. and Smith, D. E., *Plane and Solid Geometry*, p. 35. Ginn and Co. 1913.

[5] Schultze, A., Sevenoak, F. L., and Schuyler, E., *Plane Geometry*, p. 51. The Macmillan Co. 1930.

sion in substitution for one of the hypotheses of the original theorem and from the said conclusion along with the rest of the original hypothesis obtains, as its conclusion, the omitted hypothesis of the original theorem.[6]

Todhunter. In the Everyman Edition of *Euclid*, Todhunter makes the following statement:

> One proposition is said to be the converse of another when the conclusion of each is the hypothesis of the other. . . . When there is more than one hypothesis, or more than one conclusion to a proposition, we can form more than one converse proposition.[7]

Syllabus of the A.I.G.T. In the *Syllabus of Geometry* prepared by the Association for the Improvement of Geometrical Teaching, (A.I.G.T.), the following statement appears:

> Sometimes the hypothesis of a theorem is complex, i.e., consists of several distinct hypotheses; in this case every theorem formed by interchanging the conclusion and one of the hypotheses is a converse of the original theorem.[8]

Schultze. In 1912 Schultze pointed out that the commonly accepted definition of the converse "is not quite exact" and that

> . . . if the hypothesis of a theorem consists of several statements, each of them may be exchanged with the conclusion, thus giving rise to several converses, that may be, or may not be, true.[9]

But the words of Schultze fell upon deaf ears. Most of the textbooks still continued to state that the exchange of hypothesis and conclusion produced the converse of a theorem. Ironically enough the textbook that Schultze himself wrote included the very definition that he termed "not quite exact."

Christofferson. In 1933 Christofferson made a fuller analysis of the problem of defining a converse and pointed out that

> The usual definition "to interchange the hypothesis and conclusion" often leads to an absurd or impossible statement. It is only when there is but one simple condition in the hypothesis and one simple consequence in the conclusion that such interchange is possible.[10]

This is especially true when a set of three or four hypotheses yields only one conclusion. According to the traditional definition the converse of such a proposition would assert that if the conclusion is true the three or four statements in the hypothesis might follow as true consequences. Unfortunately, it has been impossible to discover even one theorem of the above description which yields a true converse. Moreover, it is highly improbable that any sane mathematician would ever formulate such a converse, let alone attempt to prove it.

Excessive Dependence upon Verbal Formulation of Theorem

As pointed out above, it was natural to infer from the phrase "the converse" that every theorem has but one converse. As a result students and teachers of geometry formed the vicious habit of stating the converse by merely paraphrasing the verbal form of the original statement without first analyzing carefully the data of the theorem. The following two theorems are considered converses of each other in the majority of textbooks examined:

> 1. If a straight line is perpendicular to a radius at its outer extremity, it is a tangent to the circle.
> 2. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

[6] Heath, *op. cit.*, Vol. 1. pp. 256–257. See also Smith, D. E. *The Teaching of Geometry*, p. 175. Ginn and Co. 1911.
[7] Todhunter, *op. cit.*, p. 255.
[8] Wilson, J. M., *Elementary Geometry*. New Edition, p. 3. London: Macmillan and Co. 1881.
The Association for the Improvement of Geometrical Teaching, (1) *Syllabus of Plane Geometry*, p. 4. London: Macmillan and Co. 1875, 1889. (2) *The Elements of Plane Geometry*. Part I, p. 13. London: W. Swan Sonnenschein and Company, 1884.
[9] Schultze, Arthur, *The Teaching of Mathematics in Secondary Schools*, pp. 144, 146. The Macmillan Co. 1912.
[10] Christofferson, H. C., *Geometry Professionalized for Teachers*, p. 123. Oxford, Ohio. 1933.

If one depends only upon the verbal formulation, the second theorem may seem on first thought to be a perfect illustration of the derivation of a converse by a complete interchange of the data in hypothesis with the conclusion. But a more careful analysis will reveal the error of such a view.

The first theorem may be analyzed as follows:

Data	Conclusion
a circle[11]	AP is a tangent
1. O is the center of the circle	
2. P is a point on the circle	
3. OPA is a right triangle	

The second theorem may be analyzed as follows:

Data	Conclusion
a circle	OPA is a right triangle
1. O is the center of the circle	
2. P is a point on the circumference	
3. AP is a tangent	

The above analysis shows clearly that the second theorem is obtainable from the first, not by interchanging the entire hypothesis and the conclusion, but rather by retaining *all the data but 3* and interchanging *it alone* with the conclusion.

Textbooks Disregard Their Own Definition

Fortunately the mathematical intuition of the writers of books was often sounder than their logical consistency. As a result they frequently named certain pairs of theorems converses of each other not in accordance with their own definition.

Thus many textbooks containing the usual definition nevertheless considered the following two theorems converses of each other:

1. If two triangles have two sides of the one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.

2. If two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.

By no possible interpretation of the traditional notion could the second theorem be considered the converse of the first. As a matter of fact the second theorem may be derived from the first by interchanging only *one* datum with the conclusion. The following analysis will clarify this point.[12]

Theorem 1	Theorem 2
Data	Data
triangles ABC and $A'B'C'$	triangles ABC and $A'B'C'$
1. $a=a'$	1. $a=a'$
2. $b=b'$	2. $b=b'$
3. $C>C'$	3. $c>c'$
Conclusion	Conclusion
$c>c'$	$C>C'$

Evidently datum 3 and the conclusion of Theorem 1 are the conclusion and datum 3 respectively of Theorem 2.

Similar statements may be made about the following pair of theorems which are generally regarded as converses of each other:

3. In the same circle, or in equal circles, equal chords are equally distant from the center.

4. In the same circle, or in equal circles, chords equally distant from the center are equal.

It will be sufficient, however, to point out that in the above pair of theorems, the verbal forms are misleading. A detailed analysis similar to that made for Theorems 1 and 2 will reveal the fact that only *one* of the conditions of the hypothesis is interchanged with the conclusion.

[11] Note the convention adopted in this study *not* to number the statement about the figure (circle, triangle, quadrilateral, parallelogram), since the data (conditions, facts) are about that figure.

[12] See note 11 in preceding column.

Many Converses Overlooked

The unsatisfactory nature of the usual definition also reveals itself in the tendency of writers to overlook the logical relation of many theorems that have as much claim to be called converses as other theorems which do receive that label.

For illustration let us take the following set of four theorems:

1. The straight line perpendicular to a radius at its outer extremity is a tangent to the circle.

2. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

3. The perpendicular to a tangent at the point of contact passes through the center of the circle.

4. The perpendicular from the center of a circle to a tangent meets it at the point of contact.

Most of the textbooks point out that the second theorem is a converse of the first without saying a word about 3 and 4. If the reader examines again the analysis of Theorem 1 on page 103 he will undoubtedly notice that 3 and 4 are derived from 1 in exactly the same manner as 2, namely, by interchanging *one* datum with the conclusion. Thus, Theorem 2 may be obtained from Theorem 1 by interchanging datum 3 and the conclusion; Theorem 3, by interchanging datum 1 and the conclusion and by making a slight change in the analysis; and Theorem 4, by interchanging datum 2 and the conclusion.

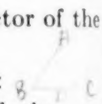
The Multi-Converse Interpretation

Proclus. As has already been mentioned on page 101, Proclus pointed out the existence of two forms of conversion, one in which the entire hypothesis and conclusion change places, and the other in which one of the conditions (or data) in the hypothesis is interchanged with the conclusion. Two criticisms may be levelled against the analysis of Proclus.

1. *First*, his terminology is inexact. The choice of descriptive terms—*conversion par excellence* for the first type and *partial conversion* for the second—gives the impression that the first type is in some way more truly a logical converse than the other.
2. *Second*, he seems to have failed to observe that in the case of a theorem in which the hypothesis consists of more than one condition it is rarely possible to form a true converse by interchanging the one conclusion and the two or more hypotheses.[13]

Perhaps this criticism can be clarified by an example. Consider the following proposition:

In an isosceles triangle the bisector of the vertex angle bisects the base.



The following is the analysis:

Data	Conclusion
a triangle ABC	$BD = DC$[14]
1. $AB = AC$	
2. AD bisects the angle A	
3. D is a point on the base	

The complete converse, or as Proclus would term it, the *converse par excellence*, would consist of the interchange of the conclusion with the three data of the hypothesis.

No elaborate proof is necessary that the truth of the three conditions in the hypothesis of the above theorem does not follow from the truth of the conclusion. Moreover, an investigation of many other theorems of a similar pattern, yielded not one case of a true "converse par excellence." Although no claim is made that such converses are never true, the following conclusion seems justifiable: Since true "complete" converses can rarely be obtained from theorems containing more than one hypothesis, it is unwise and misleading to call those propositions that can be derived "partial converses." Moreover, these "partial converses" are always meaningful and very often true.

It is interesting to note that of the three possible converses obtainable by interchanging in turn each of the three

[13] See above, page 101.
[14] See note 11 on page 103.

numbered data with the conclusion, two are true and one is false.

In textbooks. Of the textbooks examined, those mentioned in the following list stated the fact that a theorem may yield more than one converse. The order given is chronological.

Wilson, J. M. *Elementary Geometry*, p. 3. Macmillan and Company, 1881.
Halsted, G. B. *The Elements of Geometry*, p. 6. John Wiley and Sons. 1885.
Keigwin, H. W. *The Elements of Geometry*, p. 19. Henry Holt and Company. 1898.
Durell, F. *Plane and Solid Geometry*, p. 24. Charles E. Merrill Company, 1912.
Farnsworth, R. D. *Plane Geometry*, p. 51. McGraw-Hill Book Company. 1933.
Swenson, J. A. *Integrated Mathematics with Special Application to Geometry*, p. 27. Edward Brothers. 1934.
Mirick, G. R., Newell, M. J., and Harper, G. A. *Plane Geometry and Its Uses*, p. 90. Row, Peterson and Company. 1935.
Wells, W. and Hart, W. W. *Modern Plane Geometry*, p. 79. D. C. Heath and Company. 1935.

Unfortunately none of the above mentioned authors utilized consistently the multi-converse idea throughout their textbooks. Moreover, none of the textbooks fully exploited the pedagogical possibilities of that approach, especially in connection with the topics of the opposite of a theorem, locus problems, and necessary and sufficient conditions.

In manuals for teachers. Of the manuals for teachers in the English language examined by the writer, only those by Schultze and Christofferson point out the possibility of a theorem yielding more than one converse.[15] After enumerating some of the defects of the usual definition of a converse, Christofferson says:

Since the usual definition of a converse is sometimes impossible, clearly a more carefully worded one must be framed. The actual and usual conception of a converse is that a *converse* of any theorem may be stated by *interchanging any one consequence in the conclusion with any one condition given in the hypothesis.*[16]

Advantages of Christofferson's definition. Although it may be questioned whether the definition of a converse stated above is "the actual and usual conception of a converse" it is unquestionably the definition that has been used unconsciously by many mathematicians and the one that should be generally adopted with some minor modifications.

Christofferson pointed out one of the advantages of his definition in connection with the theorem

If a diameter is perpendicular to a chord, it also bisects it.

According to the usual definition, this theorem yields only one converse, which is not only false but absurd as well. On the other hand the definition that he proposes yields four propositions which "are perfectly sensible and legitimate converses."[17]

There is, moreover, another advantage which Christofferson did not point out, but which results directly from his definition, for it is applicable not only to theorems that have one conclusion but also to theorems that have two or three conclusions. Thus the theorem mentioned above frequently appears in the following form, having two or three conclusions:

If a diameter is perpendicular to a chord it bisects the chord and its subtended arcs.

Since according to the definition the converse is obtained "by interchanging any one consequence in the conclusion with any one condition given in the hypothesis," each one of the three conclusions will yield three converses. The following analysis will be carried through to make this point clearer:

Theorem A

Data	Conclusions
a circle	1. $AE=EB$
1. FD is a diameter	2. $\widehat{AD}=\widehat{DB}$
2. AB is a chord	3. $\widehat{AF}=\widehat{FB}$
3. FD is perpendicular to chord AB at E	

[15] See above, page 102.

[16] Christofferson, *op. cit.*, p. 123. The italics are in the original text.

[17] For complete details see Christofferson, *op. cit.*, pp. 123–124.

By interchanging in turn datum 1 with conclusions 1 and 2 and 3, three converses may be obtained. A similar procedure with data 2 and 3 will yield six more converses. Two of the possible 9 converses are included below.[18]

Converse I. Interchanging Datum 1 and Conclusion 1

Data	Conclusion
a circle $AFBD$	1. FD is a diameter, etc.
1. $AE=EB$	
2. AB is a chord	
3. FD is perpendicular to AB at E	

Converse II. Interchanging Datum 1 and Conclusion 2

Data	Conclusion
a circle $AFBD$	1. FD is a diameter, etc.
1. $\widehat{AD}=\widehat{DB}$	
2. AB is a chord	
3. FD is perpendicular to AB at E	

The following theorem is another interesting case where Christofferson's definition shows up to advantage:

If a figure is a parallelogram, the opposite sides are equal.

Theorem B

Data	Conclusions
quadrilateral $ABCD$	1. $AD=BC$
1. $AD \| BC$	2. $AB=DC$
2. $AB \| DC$	

The following four converses may be obtained by interchanging, one at a time, a conclusion and a hypothesis:

Converse I. Interchanging Datum 1 with Conclusion 1

Data	Conclusions
quadrilateral $ABCD$	1. $AD \| BC$
1. $AD=BC$	2. $AB=DC$
2. $AB \| DC$	

Converse II. Interchanging Datum 1 with Conclusion 2

Data	Conclusions
quadrilateral $ABCD$	1. $AD=BC$
1. $AB=DC$	2. $AD \| BC$
2. $AB \| DC$	

Converse III. Interchanging Datum 2 with Conclusion 1

Data	Conclusions
quadrilateral $ABCD$	1. $AB \| DC$
1. $AD \| BC$	2. $AB=DC$
2. $AD=BC$	

Converse IV. Interchanging Datum 2 with Conclusion 2

Data	Conclusions
quadrilateral $ABCD$	1. $AD=BC$
1. $AD \| BC$	2. $AB \| DC$
2. $AB=DC$	

The following should be noted:

1. Converses I and IV are false, whereas II and III are true.

2. Converses II and III can be expressed by the familiar theorem,

If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

None of the textbooks examined pointed out the existence of the logical relation of conversion between such two well-known theorems as Theorem B and Converse II or III. The majority of the books *do* point out, however, that the theorem,

If the opposite sides of a quadrilateral are equal, the figure is a parallelogram,

is a converse of Theorem B.

The defect in Christofferson's definition. From the point of view of the *number* of conditions (data, facts) in the hypothesis and the *number* of consequences in the conclusion, all geometric propositions may be divided into three types:

I. Theorems containing *one* hypothesis (datum, condition, fact) and *one* conclusion (consequence, result).
II. Theorems containing more than one hypothesis and only one conclusion.
III. Theorems containing more than one hypothesis and more than one conclusion.

When applied to theorems of types I and II, Christofferson's definition exhausts all possible converses that may be derived from a proposition. But when theorems of type III are considered, the definition, if exactly interpreted, does not permit the formation of many theorems that are generally called converses, and prevents the formulation of many more

[18] The form of analysis adopted here differs somewhat from the one adopted by Christofferson. Since in his analysis there are four conditions in the hypothesis, this theorem would, therefore, yield twelve converses.

theorems the logical structure of which would warrant their being termed converses of other propositions.

This defect can be illustrated by analyzing again Theorem B on page 106.

If a figure is a parallelogram, the opposite sides are equal.

Data	Conclusions
quadrilateral $ABCD$	1. $AD=BC$
1. $AD \parallel BC$	2. $AB=DC$
2. $AB \parallel DC$	

The four converses derivable from it in accordance with Christofferson's definition have already been enumerated and analyzed in the preceding section. The following proposition presents some interesting considerations:

If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

Data	Conclusions
quadrilateral $ABCD$	1. $AD \parallel BC$
1. $AD=BC$	2. $AB \parallel DC$
2. $AB=DC$	

The above proposition is generally considered a converse of Theorem B and in most textbooks it is the *only* converse associated with that theorem. Moreover, a certain verbal symmetry makes its derivation from the "mother" theorem almost irresistible. But ironically enough, on Christofferson's definition it *cannot* be considered a converse, for it is obtainable from Theorem B by interchanging the *two* data with the *two* conclusions, whereas the definition states that

A converse of any theorem may be stated by interchanging any *one* consequence in the conclusion with any *one* condition given in the hypothesis.[19]

There are, moreover, pairs of theorems which are so related that the three hypotheses and three conclusions of the one are respectively the three conclusions and the three hypotheses of the other, but which would not be called converses of each other according to Christofferson's definition. Take for illustration the following pair of theorems:

[19] Christofferson, *op. cit.*, p. 123. The word *one* is not italicized in the original text.

1. If the three angles of one triangle are equal respectively to the three angles of another triangle, the corresponding sides are in proportion.
2. If the sides of two triangles are in proportion, the corresponding angles are equal.

Since the basic idea of the process of conversion is the *interchange* of elements of the hypothesis and the conclusion, it seems capricious to permit the application of the label of converse to sets of theorems in which only *one* element has been interchanged, but refuse that appellation when two elements or when the *entire* hypothesis and conclusion have been interchanged.

The above *a fortiori* argument is in itself of sufficient force to warrant a modification of Christofferson's definition. There are, moreover, other cogent arguments for such a change, as well as advantages to be derived from it. But these will have to wait until the enunciation of the modified definition.

Converse Redefined

The following substitute definition of a converse is therefore presented:

The converse of a theorem may be obtained by interchanging *any* number of conclusions with *an equal number* of hypotheses.

This definition remedies the shortcomings pointed out above. The use of the word "any" enables it to include within its scope also those converses that are obtained by the interchange of *one* hypothesis and *one* conclusion.

Since the definition of "converse" is no more than a declaration of intention to use that word instead of a longer statement, it really needs no justification. The advantages of this definition will be pointed out below. It may not, however, be out of place to state here the *reason* for the phrasing of the definition so as to refer only to an interchange of an *equal* number of hypotheses and conclusions: *No theorem in geometry was found which yielded a true converse by an unequal interchange of hypotheses and conclusions.* Al-

though this statement is merely an empirical generalization based on a relatively small sample and therefore likely to be overthrown by the discovery of one exception, it is of sufficient importance to be used as a *guide* in the choosing of a definition.

Some Consequences of the Proposed Definition of Converse

Increase in the Number of Converses

The most obvious effect resulting from the adoption of the proposed definition will be the increase of the number of converses to which every theorem will theoretically give rise. As an outcome of the traditional notion, every theorem has but one converse, no matter how many conditions are present in the hypothesis, nor how many consequences are derived in the conclusion.

On Schultze's analysis, a theorem with three data and one conclusion will yield three converses by interchanging any one of the data with the conclusion. He did not consider in his discussion theorems having more than one conclusion. Perhaps he intended such theorems to be subdivided into their constituent theorems.

On Christofferson's definition a theorem with one hypothesis and one conclusion will yield one converse; a theorem containing a number of hypotheses and one conclusion will yield as many converses as there are hypotheses; while a theorem having m hypotheses and n conclusions will give rise to mn converses.

The definition proposed in this study increases still more the number of possible converses that a theorem may have, if it has more than one conclusion. This claim can be upheld by analyzing the following schematized theorem, having three hypotheses and two conclusions:

$$h_1h_2h_3 \rightarrow c_1c_2$$

where h represents a hypothesis, and c represents a conclusion.

The possible converses are as follows:

By interchanging one conclusion with one hypothesis:

1. $c_1h_2h_3 \rightarrow h_1c_2$
2. $h_1c_1h_3 \rightarrow h_2c_2$
3. $h_1h_2c_1 \rightarrow h_3c_2$
4. $c_2h_2h_3 \rightarrow c_1h_1$
5. $h_1c_2h_3 \rightarrow c_1h_2$
6. $h_1h_2c_2 \rightarrow c_1h_3$

By interchanging both conclusions with two hypotheses:

7. $c_1c_2h_3 \rightarrow h_1h_2$
8. $c_1h_2c_2 \rightarrow h_1h_3$
9. $h_1c_1c_2 \rightarrow h_2h_3$

If a theorem contains three hypotheses and three conclusions, the theoretical number of independent converses may be calculated as follows:

By interchanging one hypothesis with one conclusion, we obtain

$$3 \times 3 = 9$$

By interchanging two hypotheses with two conclusions, we get

$${}_3C_2 \times {}_3C_2 = 3 \times 3 = 9$$

By interchanging the three conclusions with the three hypotheses, we get

$${}_3C_3 \times {}_3C_3 = 1 \times 1 = 1$$

The total number of converses $= 19$

If a theorem contains four hypotheses and three conclusions, the number of possible converses may be calculated as follows:

By interchanging one hypothesis with one conclusion

$${}_4C_1 \times {}_3C_1 = 4 \times 3 = 12$$

By interchanging two conclusions with two hypotheses

$${}_4C_2 \times {}_3C_2 = 6 \times 3 = 18$$

By interchanging the three conclusions with any three hypotheses

$${}_4C_3 \times {}_3C_3 = 4 \times 1 = 4$$

The total number of converses equals 34.

Verbal Statement Not a Basis for Formulation of Converses

One of the undesirable results of the traditional definition of the converse was that it tended toward a slipshod, unthinking, almost mechanical way of making the

transformation. It was furthermore assumed that the mere verbal statement of the proposition was sufficient for the formation of the converse without the detailed analysis of its meaning in terms of a diagram, and without the implicit and explicit hypotheses and conclusions. It is hoped that the proposed definition of converse, which potentially attributes to every theorem at least as many converses as there are conclusions will discourage such glib logical transformations.

Exercising Mathematical Imagination

The mere numerical increase of converses would by itself be of little importance. The method of deriving converses that is advocated here has, in addition, the advantage of enabling even a beginning student to exercise his imagination and to discover theorems that he could not otherwise hit upon, except accidentally. As an illustration consider the following theorem:

The bisector of the vertex angle of an isosceles triangle is perpendicular to the base and bisects the base.

An analysis in terms of the diagram yields the following hypotheses and conclusions:

Theorem C

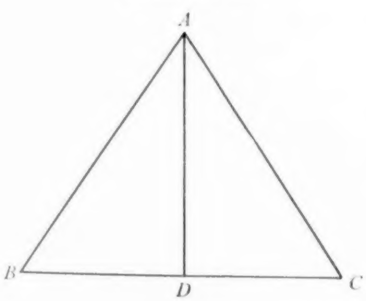


Fig. 1.

Data	Conclusions
ABC is a triangle	1. $BD=DC$
1. $AB=AC$	2. $AD\perp BC$
2. AD bisects angle A	

According to the calculation on a preceding page, five converses may be obtained from the above theorem.

The five converses follow:

Converse I

Data	Conclusions
ABC is a triangle	1. AD bisects angle A
1. $AB=AC$	2. $AD\perp BC$
2. $BD=DC$	

Converse II

Data	Conclusions
ABC is a triangle	1. AD bisects angle A
1. $AB=AC$	2. $BD=DC$
2. $AD\perp BC$	

Converse III

Data	Conclusions
ABC is a triangle	1. $AB=AC$
1. $BD=DC$	2. $AD\perp BC$
2. AD bisects angle A	

Converse IV

Data	Conclusions
ABC is a triangle	1. $AB=AC$
1. $AD\perp BC$	2. $BD=DC$
2. AD bisects angle A	

Converse V

Data	Conclusions
ABC is a triangle	1. $AB=AC$
1. $BD=DC$	2. AD bisects angle A
2. $AD\perp BC$	

The following points should be noted in this connection:

1. All the five converses are true. This is not always the case as the examples later in this section will show.
2. In order to prove these converses different techniques are required. Although their proofs are not all within the range of the beginning student, their formulation is in itself a worthwhile excursion into mathematical exploration.

Another interesting example of discovery of propositions by means of the revised concept of converse follows:

Data	Conclusions
the curve $AFBD$ is a circle	1. $AE=EB$
	2. $\widehat{AD}=\widehat{DB}$
1. AB is a chord	3. $\widehat{AF}=\widehat{FB}$
2. FD is a diameter	
3. $FD\perp AB$ at E.	

In accordance with the analysis on p. 108 the above proposition yields nineteen converses.

Although any other theorem having more than one conclusion will yield converses which have not occurred to many beginning students of geometry, the following one is of special interest:

The opposite sides and opposite angles of a parallelogram are equal.

Theorem D

Data	Conclusions
quadrilateral $ABCD$	1. $AB=DC$
	2. $AD=BC$
1. $AB \| DC$	3. angle A = angle C
2. $AD \| BC$	4. angle B = angle D

Without attempting to enumerate all the possible converses, the following facts should be of interest:

1. The number of conclusions is greater than the number of data.

2. Some of the converses are true, while others are false.

I. True Converse

Data	Conclusions
$ABCD$ is a quadrilateral	1. $AB \| DC$
	2. $AD \| BC$, etc.
1. $AB=DC$	
2. $AD=BC$	

The verbal statement of the above is the well-known theorem

If the opposite sides of a quadrilateral are equal, it is a parallelogram.

II. False Converse

Data	Conclusions
$ABCD$ is a	1. $AB=DC$
quadrilateral	2. $AD \| BC$
1. $AB \| DC$	3. angle A = angle C
2. $AD=BC$	4. angle B = angle D

The above theorem is evidently false when $ABCD$ is an isosceles trapezoid. Nevertheless, this converse is of pedagogic importance, for it affords an excellent transition from the topic of parallelograms to that of trapezoids.

3. The theorem,

If the opposite sides of a quadrilateral are equal, the figure is a parallelogram,

is recognized and treated by textbook writers as a converse of the theorem

The opposite sides of a parallelogram are equal.

Oddly enough none of the books examined recognize the theorem below as also a converse of the same theorem.

If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

Data	Conclusion
$ABCD$ is a quadrilateral	1. $AD \| BC$, etc.
1. $AB \| DC$	
2. $AB=DC$	

Obviously the above theorem may be obtained from the original theorem in the first part of this section by interchanging datum 2 with conclusion 1.

This illustration is only one of many disguised converses that abound in geometry. But that topic is so significant as to warrant treatment in a separate section.

Disguised Converses

Theorems of congruence. Perhaps the most important consequence of the adoption of the proposed definition of converse is the realization that many more theorems are logically related to each other than might be suspected. The last theorem mentioned in the preceding section was an example of such a relation. But the logical interconnection between the various theorems of congruence is too interesting not to be specifically analyzed.

Let us take the theorem which is usually the first of the set in most textbooks:

Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

The analysis in terms of data and conclusion may be stated as follows:

Theorem E

Data	Conclusions
triangles ABC and $A'B'C'$	1. $A=A'$
	2. $B=B'$
1. $a=a'$	3. $c=c'$
2. $b=b'$	
3. $C=C'$	

The following are only a few of all possible converses:

Converse I. Interchanging Datum 3 with Conclusion 3

Data	Conclusions
triangles ABC and $A'B'C'$	1. $A=A'$
1. $a=a'$	2. $B=B'$
2. $b=b'$	3. $C=C'$
3. $c=c'$	

The above converse is the theorem

Two triangles are congruent if three sides of one are equal to three sides of another. (*s.s.s.* = *s.s.s.*)

Converse II. By Interchanging Datum 1 with Conclusion 1

Data	Conclusions
triangles ABC and $A'B'C'$	1. $a=a'$
1. $A=A'$	2. $B=B'$
2. $b=b'$	3. $c=c'$
3. $C=C'$	

The reader has undoubtedly recognized in the foregoing data and conclusions the theorem

Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other. (*a.s.a.* = *a.s.a.*)

Converse III. By Interchanging Datum 2 with Conclusion 1

Data	Conclusions
triangles ABC and $A'B'C'$	1. $b=b'$
1. $a=a'$	2. $B=B'$
2. $A=A'$	3. $c=c'$
3. $C=C'$	

The above is the theorem

Two triangles are congruent if two angles and the side opposite one of them are equal respectively to two angles and the corresponding side of the other. (*s.a.a.* = *s.a.a.*)

These three converses are all true, and require, of course, individual proofs. But none of the textbooks examined pointed out that such an interlocking logical relation exists between them.

Not only are the above true theorems related to the original proposition, but the "ambiguous case," too, can be shown to be a converse of the same theorem by the following analysis:

Converse IV. Interchanging Datum 3 with Conclusion 1

Data	Conclusions
triangles ABC and $A'B'C'$	1. $C=C'$
1. $a=a'$	2. $B=B'$
2. $b=b'$	3. $c=c'$
3. $A=A'$	

If the proposed definition is accepted, the following theorem may also be treated as a converse:

If the three angles of one triangle are equal respectively to the three angles of the other, the corresponding sides are equal.

This false proposition, so alluring to the beginning student of geometry, may be obtained from the original Theorem E on the preceding page, as follows:

Converse V. Interchanging Data 1 and 2 with Conclusions 1 and 2

Data	Conclusions
triangles ABC and $A'B'C'$	1. $a=a'$
1. $A=A'$	2. $b=b'$
2. $B=B'$	3. $c=c'$
3. $C=C'$	

Disguised converses among theorems of similarity. The following analysis, similar to the one given above for the theorems of congruence, indicates that the three theorems dealing with conditions of similarity of triangles are also bound together by the logical tie of conversion, if that transformation is interpreted in the way advocated here. The basic theorem on the similarity of triangles is

Two triangles are similar if two angles of one are equal respectively to two angles of the other.

The analysis of the above theorem is as follows:

Theorem F

Data	Conclusions
triangles ABC and $A'B'C'$	1. $a:a'=b:b'$
1. $A=A'$	2. $a:a'=c:c'$
2. $B=B'$	

By interchanging data 1 and 2 with conclusions 1 and 2, the following converse is obtained:

Converse I

Data	Conclusions
triangles ABC and $A'B'C'$	1. $A=A'$
1. $a:a'=b:b'$	2. $B=B'$
2. $a:a'=c:c'$	

Put into words, Converse I becomes the familiar theorem

Two triangles are similar if their corresponding sides are proportional.

The textbooks examined do not indicate the logical relationship between Converse I and Theorem F.

Another converse may be obtained from Theorem F by interchanging datum 1 with conclusion 2.

Converse II

Data	Conclusions
triangles ABC and $A'B'C'$	1. $a:a'=b:b'$
1. $a:a'=c:c'$	2. $A=A'$
2. $B=B'$	

The verbal equivalent of Converse II is

Two triangles are similar if an angle of the one is equal to an angle of the other and the sides including these angles are proportional.

Needless to say the logical relation between Converse II and the original theorem was also disguised by an incomplete analysis of the meaning of the term *converse*.

More interesting than either Converse I or Converse II is the following proposition which may be obtained from the original theorem by interchanging datum 2 with conclusion 1:

Converse III

Data	Conclusions
triangles ABC and $A'B'C'$	1. $a:a'=c:c'$
1. $A=A'$	2. $B=B'$
2. $a:a'=b:b'$	

Converse III may be formulated as follows:

Two triangles are similar if an angle of the one is equal to an angle of the other, and the ratio between the opposite and an adjacent side of the one is equal to the ratio of the corresponding sides of the other.

Obviously this theorem is not always true and corresponds to the ambiguous case in congruent triangles. It is interesting to note that the conditions for similarity in the ambiguous case are equivalent to those for congruence. The following analysis will make this clear:

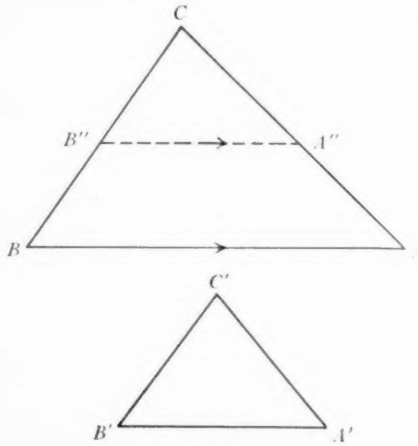


Fig. 2.

Data	
triangles ABC and $A'B'C'$	To find out under what conditions the triangles are similar.
1. $A=A'$	
2. $CB:C'B'=CA:C'A'$	

Analysis

1. On CB lay off $CB''=C'B'$
2. Construct $B''A''$ parallel to BA
3. $CB:CB''=CA:CA''$
4. But $CB''=C'B'$
5. Therefore $CA''=C'A'$
6. angle $CA''B''=CAB=C'A'B'$
7. Therefore triangle $C'B'A'$ and triangle $CB''A''$ have two sides and angle opposite one of the sides equal respectively etc.
8. Since triangle $CB''A''\sim$triangle CBA, the same conditions that determine the congruence of $C'B'A'$ and $CB''A''$, will also determine the similarity of triangle $C'B'A'$ and triangle CBA.

Without going into further analysis it is evident from the above diagram that if $CB''<CA''$, there is another point B''' (not shown) on $B''A''$ so that $CB''=CB'''$. The triangle $CB'''A''$ (not shown in the figure) will not be similar to $CB''A''$ and therefore not similar to CBA.

Disguised converses in solid geometry. Because of the greater complexity of the subject matter of solid geometry there are many propositions that pass unrecognized as the converses of previously established theorems. The examples below will illustrate this point:

The two following theorems are usually proved on adjoining pages of many textbooks without the logical connection being pointed out.

I. Planes perpendicular to the same straight line are parallel to each other.
II. A straight line perpendicular to one of two parallel planes is perpendicular to the other also.

Theorem I may be analyzed as follows:

Data	Conclusion
1. Plane 1 is perpendicular to line AB	Plane 1 is parallel to plane 2
2. Plane 2 is perpendicular to line AB	

Theorem II may be analyzed as follows:

Data	Conclusion
1. Plane 1 is parallel to plane 2	AB is perpendicular to plane 1
2. AB is perpendicular to plane 2	

Evidently Theorem II is the converse of I, for it may be obtained from it by interchanging datum 1 with its conclusion.

Summary

In this chapter the concept of the converse of a proposition was analyzed. The defects of the traditional definition were pointed out as well as the shortcomings of those offered by Schultze and Christofferson. A substitute definition was proposed which not only does not have the defects mentioned above, but possesses some positive advantages, both logical and pedagogical, to recommend its adoption.

(*To Be Continued*)

Ratio as Multiplier*

By George W. Evans
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Of course a ratio can be defined either as a quotient or as a multiplier. At a recent meeting of this Association I heard a distinguished scholar say that the notation $y \propto x$ should be regarded as obsolete and replaced by the equation $y = mx$. It is the equivalent of the orthodox statement "y is proportional to x," and you notice the ratio is the constant multiplier m.

The phrase "the ratio of A to B" originated in attempts to translate Euclid's definition of proportion. He defined *ratio* as "a *schesis*, or such, depending on measurement." The word *schesis* means literally "control" and the idea is that it controls the value of A when B is given. In technical use it was confined to rational fractions used as multipliers. The word translated by the phrase "or such" was meant to allow the use of multipliers which were not rational. The word which I have translated as "measurement" is *pelikotes*, literally "how-much-ness," in contrast with the word *posotes*, which means "how-many-ness." With these things in mind we can translate Euclid's definition as follows: "The ratio of A to B is the measurement number of A when the unit is B." Further, Euclid's definition of proportion contains a careful description of the process of measurement by an undivided unit.

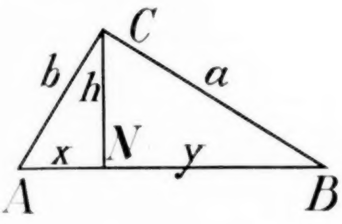
By the expression "measurement number" I mean the number which must go with the unit to give the measurement of a quantity. Every measurement includes two things: the unit of measure, and the measurement number. For example, if the speed of an automobile is fifteen miles an hour, that is the same as twenty-two feet a second. The unit in the first case is a speed of one mile per hour, and the measurement number is 15; the unit in the second case is one foot per second, and the measurement number is 22. The same speed must have different measurement numbers if different units are used. A meter is 39.370 inches long. The 39.370 is a "*schesis*." It is the measurement number of the meter in terms of the inch. It is the definite idea back of the words "the ratio of the meter to the inch."

As a matter of fact the trigonometrical ratios, however defined, are generally used as multipliers. The tables contain not only the sine, but also the cosecant; not only the cosine, but the secant; not only the tangent, but the cotangent. It is an indication of the hatred that mankind has always felt for long division. It goes back forty centuries. You remember Professor Archibald told us that on the bricks of ancient Babylon there were tables of reciprocals in their cumbersome sexagesimal notation; rather than divide they hunted up these reciprocals and multiplied.

In the textbooks of trigonometry or surveying you will find a problem about the engineer who came to a turbulent river and wanted to find the distance from where he stood to a tree on the opposite bank. He sent his red-headed rodman down along the bank a distance which turned out to be 82.35 feet, and measured the angle between the rodman and the tree as 75°; then he changed places with the rodman and measured the other angle between the rodman and the tree as 45°. Being good at arithmetic he was able to say that the third angle in the triangle was 60°. Now he had to divide 82.35 by the sine of 60°, which is .8660, and multiply the result by the sine of 45°, which is .7071. Well, my guess is that he did not divide by .8660, but multiplied by its reciprocal, the cosecant, which is 1.1547.

* An address delivered before the *Association of Teachers of Mathematics in New England* on Dec. 4, 1937.

The familiar hypotenuse theorems furnish a good illustration of the use of cosines as multipliers. In this figure the angle ACN is equal to B and the angle BCN is equal to A. And we have the following equations:



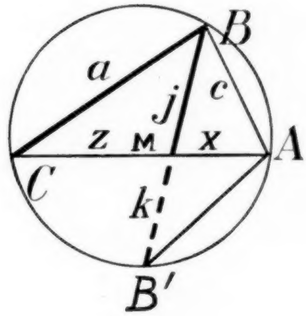
(1) $a = c \cos B$; $b = c \cos A$
(2) $x = b \cos A$; $y = a \cos B$
(3) $h = a \cos A$; $h = b \cos B$.

From these we get by substitution $x = c \cos^2 A$, and $y = c \cos^2 B$; then $x + y = c \cos^2 A + c \cos^2 B = c$.
Multiplying through by c, $c^2 \cos^2 A + c^2 \cos^2 B = c^2$, or $a^2 + b^2 = c^2$. The first term of this equation is b^2, but it is also cx; the second term is a^2, but it is also cy. That gives us another one of the hypotenuse theorems. Another one, $h^2 = xy$, can be obtained by multiplying the two values of h in (3), and then the values of x and y in (2).

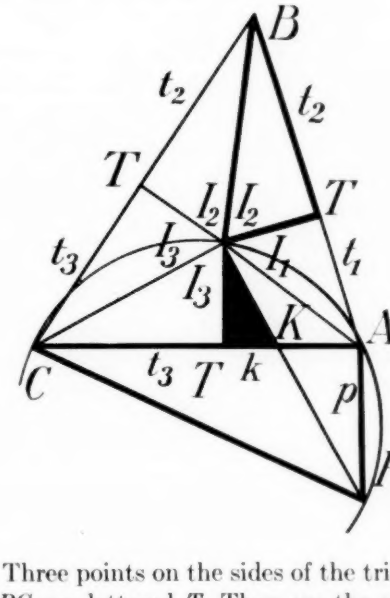
The two equations $y = mx$, $b = ma$, replacing the proportion $y:x = b:a$, can be handled quite as well as the "means and extremes" of the old notation. You need only to multiply equals by equals so that m comes on both sides of the resulting equations: $may = mbx$. Like the sacred Rule of Three, it is a trick; but isn't it a good one?

The formula for the length of a bisector used to be in every textbook of geometry, but is not so cherished now. The proof here given uses single letters for the lengths of lines,—another change from the days when an "algebraic proof" was abhorrent. The triangle $BB'A$ is similar to BCM.

(1) Let $j + k = ma$; then $c = mj$.
(2) $mj(j + k) = mac$; $ac = j^2 + jk$.
(3) But $jk = zx$; so $ac = j^2 + zx$.



The formula for the area of a triangle in terms of the sides, though it is called Heron's formula, is now regarded as the invention of Archimedes. Anyway, here is Heron's proof of it, with his own diagram. The lettering of the diagram is modified, and the ratios are utilized as in the preceding discussions; as a result the demonstration is much easier to follow.



Three points on the sides of the triangle ABC are lettered T. They are the points of contact of the incircle, whose center is I. Thus there are three perpendiculars IT, each equal to r. The two rt tgls ATI are congruent, and taken together from a right-angled kite, two of whose sides are equal to r, and the other two sides are each t_1, the tangent to the incircle from A.

The area of the kite $ITAT = rt_1$; that of $ITBT = rt_2$; that of $ITCT = rt_3$. Then

the total area of ABC is $r(t_1+t_2+t_3)$.

The side $a=t_2+t_3$; $b=t_3+t_1$; $c=t_1+t_2$. Then $a+b+c=2(t_1+t_2+t_3)$ and $s=t_1+t_2+t_3$. Moreover $s-a=t_1$; $s-b=t_2$; $s-c=t_3$; and the area of $ABC=rs$. What we have to prove then is $r^2s^2=s(t_1t_2t_3)$.

(1) Draw perpendiculars, one to CI at I and another to CA at A, and mark their intersection P. The circle on CP as diameter will be the ccmcircle of CIA; and angle $CPA=180°-CIA$.

(2) The diagonal of each kite bisects the angle at I, so that $2I_1+2I_2+2I_3=360°$; then CIA, which equals I_3+I_1, is the supplement of I_2; then angle $CPA=I_2$, and tgl $ACP\sim TBI$.

(3) Let $t_3+t_1=mt_2$; then $p=mr$; and $s=t_1+t_2+t_3=(m+1)t_2$.

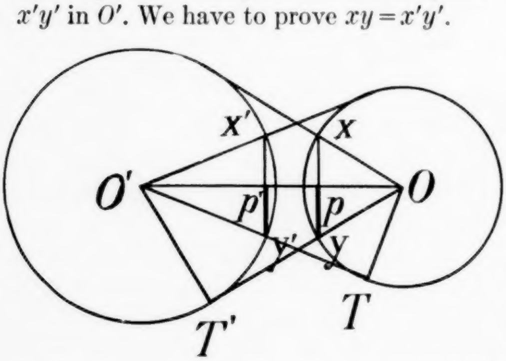
(4) Tgl $TKI\sim AKP$; then $t_1-k=mk$, and $t_1=(m+l)k$.

(5) $(m+1)ks=(m+1)t_1t_2$; $ks=t_1t_2$.

(6) In the right tgl CIK, $kt_3=r^2$, and $k=r^2/t_3$.

(7) Substituting (6) in (5), $r^2s=t_1t_2t_3$; and $r^2s^2=st_1t_2t_3$.

Next is a problem harder to describe than to solve. It came from some examination paper, and is very neatly dealt with by using the ratio of similarity as a multiplier. The diagram shows two circles, O and O', with tangents drawn to each from the center of the other. Those tangents intercept a chord xy in the circle O, and $x'y'$ in O'. We have to prove $xy=x'y'$.



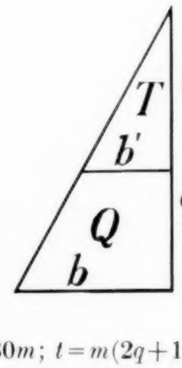
We take the half-chords, p in O and p' in O', and let $OO'=d$. There are two right triangles with the angle $O'OT'$, which are therefore similar. One has the hypotenuse d, the other r.

(1) Let $r=md$, then $p=mr'$.

(2) In the same way, let $r'=kd$, then $p'=kr$.

(3) Substituting for r' and for r, $p=mkd$, $p'=kmd$. There you are.

Going back to Babylon we find on their tablets of baked clay several problems about right triangles which are divided by a line (call it b') parallel to one of the legs (call it b) into a triangle (area T) and a quadrilateral (area Q), with altitudes t and q respectively. Taking one of those problems as a sample, Q is given as 480, b is 30, and q is 10 less than t. Since T is similar to the whole triangle $T+Q$, and the altitude of $T+Q$ is $t+q$, or $2q+10$, we can let $m=$ the ratio of similarity of the triangles, and then



(1) $b'=30m$; $t=m(2q+10)=q+10$.

(2) $q(2m-1)=10(1-m)$, and $q=10\ (1-m)/(2m-1)$.

(3) $b'+b=30(m+1)$; $Q=480=15\ (m+1)10(1-m)/(2m-1)$.

(4) $16=5(m+1)(1-m)/(2m-1)$; $32m-16=5-5m^2$; $5m^2+32m-21=0$.

(5) $(5m-3)\ (m+7)=0$, and $m=3/5$.

From that result all the other details about the figure can easily be computed.

The Wise Men of Babylon did not take such a direct and simple way to the solutions of such problems; but they showed that they knew about similar right triangles, and they walked doggedly through the successive steps of the quadratic formula, wherever they got it. Moreover, out of their simple diagram more than 40 different problems can be made, all leading to quadratics.

On the Need of Cooperation Between High School and College Teachers of Mathematics*

By AUBREY KEMPNER, *President of the Mathematical Association of America*
University of Colorado, Boulder, Colorado

IT IS an honour to be invited to speak to you. Some months ago I talked to my colleagues in the collegiate field about the incredible difficulties in the present high school situation, and pleaded for their cooperation in any form you may desire.[1] To you, I need not say much concerning the facts. You are more keenly aware of them than I can be. It would only be carrying owls to Athens. Instead, I shall attempt a constructive suggestion, without unfriendly criticism of any kind.

One cannot teach a subject a quarter of a century without coming into contact with all pedagogical ramifications of the field. After about ten years one realizes dimly what ten minutes of clear thinking would have taught one: The secondary school system and the colleges and universities are organically intertwined. Our college students are the future teachers of high school boys and girls; and these boys and girls will come to the universities as students.

Anything that I may say can be only a recognition of this fundamental fact and its consequences. We cannot consider either the high school situation or the college situation in a vacuum, as it were. Of course the high schools have their own problems, and the colleges have theirs, and each group may be assumed capable of running its own affairs without outside interference. If university professors criticize high school teachers, they might do well to remember that every one of these teachers has received the stamp of approval of a college or a university.

On the other hand, sweeping criticism of university methods, or of high school methods, should have no room among us. Every fault in the secondary system I will undertake to compare with an equally glaring fault in the college system; and many of you will admit that you can match every evil in the college with one of the peculiar devils that haunt you.

Without hope of contradiction I may say that we have a deplorable lack of coordination and cooperation between high school and college teachers. There is a discouraging record of opportunities missed.

Following the steps of E. H. Moore, J. W. Young, D. E. Smith, your late honorary president H. E. Slaught, Earle Hedrick and others we have an imposing list of men, some of whom are present in this assembly, equally prominent as trained mathematicians and as experts in the teaching of mathematics. Their number is not small and is increasing. But, numerically, they form a minute fraction of our two groups. They are performing valuable services, but I do not believe that their work has the necessary resonance which can be supplied only by a widespread interest in their aims and cooperation in their work. I make an earnest plea for serious and friendly consideration among both high school and college teachers of the efforts of these men who are primarily mathematicians.

Taking everything into account, may we claim that very many among us think of ourselves, not primarily as high school teachers or college teachers, but as teachers of mathematics, as parts of a larger whole? We are all engaged in the same task, and the community of our interests must always be before our eyes.

[1] See *American Mathematical Monthly*, vol. 44, December 1937.

* An address given by invitation at the meeting of the National Council of Teachers of Mathematics at Indianapolis, December 1937.

We must endeavour to see clearly the common elements in our problems; and on these common tasks our best combined efforts must be brought to bear.

First and foremost among these common problems is the question of continuity of mathematical instruction through high school and college.

It will not do—as we are finding out at a price—for universities to adopt a take or leave it attitude concerning college prerequisites and entrance requirements. Outside the realm of relativity, the tail cannot wag the dog. I do not wish to imply that universities should fail to fight for their principles and convictions; but they should offer considered reasons for their attitude. They should be willing and anxious to discuss their position with the high school group; but they should defend it only in so far as they can offer valid arguments. The university must reconsider its historical attitude that the main purpose of the high school is the preparation of future university students.

No more than ten per cent of the high school population continue to the university. If I were engaged in high school teaching, I should have little patience with a flat dictum that all of my students should be prepared as if they were going to college. This attitude has caused much trouble and will cause much more, if not checked.

But equally hard to understand is the failure of vast numbers of high school people to recognize that the preparation of future students is, and should be, a fundamental object of the high school. If our student population does not receive a satisfactory training in high school, if the university must assume in its first year, or in its first two years, the functions of the high school, the effects will be incalculable. We all know how far we have already traveled along this road.

It is easy to say that the university should reach lower down and that it is perfectly proper to devote the first year or two of college to work traditionally considered within the domain of the secondary system. There are many to whom it seems desirable to enlarge the four year curriculum to five or six years in order to allow the inclusion of these matters. But many university teachers are convinced that already under present conditions four years are quite inadequate to furnish a cultural training as well as a specific training in some chosen field. Those who favour the inclusion of ever more high school work in the university ignore one aspect which has received but scant consideration. I shall not discuss the economic waste of such plans, but shall state my conviction that most high school subjects are characterized by an optimum age; that such subjects as beginning languages, history, much of elementary mathematics, should be taught during high school years; that it is a waste of time and energy to postpone to the age of eighteen or nineteen work that can be more easily performed in the formative years from thirteen to sixteen. For reasons beyond our control, the freshman level in the country as a whole has always been unreasonably low. We must turn to countries which we consider uncivilized to find as low a level for university entrance as is prevalent in the United States.

But if we on the one hand admit that the preparation of students for college is only one of the aims of the high school and on the other hand insist on proper preparation of the student graduating from the secondary system into college, we must consider with an open mind the efforts of the high school to discriminate between students intending to continue and those for whom the secondary training is terminal. If the secondary school system is able to present a program which will give to the one group an integrated education providing them with the essentials for a rich and useful life, while supplying the other group with a thorough foundation for university work, it will not do for the college people to turn it down on the basis of pre-conceived ideas; rather, it

must be given careful and sympathetic consideration.

Such differentiation of courses according to ultimate aims—which will, on the whole, imply a differentiation according to ability—is justified. In this country where the entering high school population is, at least theoretically, nearly unselected, common-sense tells us that undifferentiated standards must work hardships on the ungifted, and have a paralyzing effect on the mentally alert and talented. High schools are without question justified in reorganizing their curricula.

Even if the high schools were turning out students adequately prepared in mathematics for college, it would still be necessary to admit a more elastic arrangement than the present one.

But what are the actual results? Do we, either in the university or in high school, give our students the minimum amount of mathematics required for dealing intelligently with everyday problems? I am at present not speaking of cultural claims, which have been presented by such leaders as President Hutchins of the University of Chicago far more ably than I could hope to do, but only of utilitarian considerations.

A member of congress says: "The South has reduced the crime of lynching more than 5000% in the last 40 years." A headline proclaims: "Italy reclaims half a billion acres of swampland in 14 years." A small matter of confusing billions and millions, which is persistently continued through half a column! However, when dealing with very large numbers, is it unreasonable to suspect that many of our national leaders see no difference between 1,000,000,000 and 10,000,000,000, except that one of the numbers has a zero more than the other—and zero means nothing. (That zero means nothing I learned when a student told me: "5 divided by zero is zero!" His explanation was: You said in class, "5 divided by zero is meaningless; it means nothing at all"; and nothing at all is zero.) From the Literary Digest: "In his more or less uneventful life, he has established a 1000% success score."

Sometimes the lack of understanding of mathematics is capitalized by industrial concerns in very misleading fashion. In an advertisement for car-financing I find: $624.—to be borrowed.—6% financing charge: $37.44.—Total: $661.44.—Payable in 12 monthly installments of $55.12. This works out at about 11% rate of interest.

One of the purest gems is contained in a report on Russia's state of naval preparedness; "That Russia was amply prepared in the event of war was indicated by the announcement her naval forces had increased 1700% since 1933.—Admiral Ivan Orloff, chief of naval forces, listed the following increases: Submarines 715%; warships 300%; coast artillery 75%; anti-aircraft guns 100%; marine aviation 510%."—It adds up correctly to 1700.

This reminds me of my wife's arithmetic. When the banks closed in the depression, we were caught and had to waive 25%. With the acumen of a true mathematician I transferred to another bank which also closed and asked for a 25% waiver. My wife's comment was; "Now see what you've done. You lost 25% in the first bank, and 25% in this one; together, 50%."

Or, consider the financial mess in which my own state, Colorado, finds itself. Unless something radical is done many important state institutions such as hospitals and state colleges may have to seriously curtail their activities. This situation is caused by an amendment to the state constitution, passed by a referendum of the people, awarding a $45-monthly pension to all "needy" citizens of Colorado from the age of 60 years on. An annuity of that type and amount costs over $7000—, so that, what the state has voted upon itself, as a consequence of general ignorance of elementary arithmetic of finance, is essentially to make a gift of over $7000— to every man and woman of age 60, to be paid from public taxes.

No, even from a narrow utilitarian standpoint our present system is quite inadequate, both in high school and in college.

To illustrate what we accomplish, or fail to accomplish, in other than utilitarian aspects, I mention only one comment by a student in the University of Colorado after taking analytic geometry. He said to his instructor: "Now if this were a course like English or history, there would be some logic in the course, but as it is, it is just memorizing formulae and working problems. There's no logic in it."

This student had taken at least one year of high school geometry and one year of high school algebra, and two quarters of University of Colorado mathematics before entering the course which inspired his comment.

The gradual recession of mathematics in the high school is vividly brought out in a pamphlet concerning the old Franklin High School in Philadelphia: "Opening in 1826, to 304 pupils, the courses offered and their distribution was: English 300; French 153; Latin 105; Greek 35; Spanish 45; German 20; elocution 300; geography 240; drawing 231; and 'all of them mathematics'."

One cannot avoid the impression that the reaction of responsible authorities to the existing situation has not been: "Our mathematics teaching has failed to fulfill its purposes; therefore we must make all efforts to improve it," but rather: "The results show that the time spent on mathematics is largely wasted; therefore, reduce mathematics to a minimum; eliminate everything beyond ordinary arithmetic and such simple geometrical situations as occur in every day life."

Much as this attitude must go against the grain of a mathematician, we must credit its protagonists and defenders with good faith. We must remember that they all had a standard mathematical training in their high school or college career, and that it is our teaching which has left them with their damning opinion of the value of mathematics. (It has been reserved to the present day to make it possible for a student to graduate from a first-class university, to continue to his master's degree and to his doctor's degree, with no mathematics at all beyond the grades.)

But after acknowledging our faults and admitting our failures, we claim the right to examine critically all proposals. We must fight mistaken conclusions, and we must insist that all mathematics teachers of the country, through their organizations, representing all levels of mathematical instruction, shall have a determining voice in the question of mathematics in our whole system of education.

I have stated that, however much we may disagree with any plans proposed, we must assume that they are the outcome of honest but misguided thinking. Their proponents are convinced that they are offering a panacea, and, just as we criticize their lack of understanding of the fundamentals of the situation they are puzzled by our reactionary attitude which seems to ignore the signs of the time.

What are the changes proposed for the high schools?

I am considering only the broadest outlines. I ignore such questions as unified or compartmental courses in high school mathematics. This is a fundamental problem of detail but involves no vital principle. With proper teachers either system should be workable. With this understanding, all proposed changes seem to fall into two large categories. They have in common that each of them further threatens the weak position of mathematics in the secondary system, unless subjected to material modification.

The one school of thought would leave the formal position of mathematical courses in the high school essentially undisturbed, perhaps eliminating the advanced courses, but "socializing" what is left by less emphasis on formal proof and by greater stress on applications as they arise in the daily life of the average citizen.

At the same time, the courses may be made elective.

The other group of reformers would do away entirely with mathematics as a subject and introduce such portions of mathematics as fit into some integrated "project" plan as the need arises. (I am not using the term "project" plan in any technical sense, but only to express a certain trend.)

It cannot be denied that each of these two major plans for reform offers something of importance.

The first, to retain most of the formal courses, but to dilute them and make them largely elective, might conceivably serve the purposes of the majority of those who do not continue beyond high school; but unless supplemented by appropriate mathematics courses for those who will continue, the colleges of the country will face an impossible situation.

Here I must emphasize a difficulty which affects the vast sparsely populated portions of the country much more than the East, Middle West, and Pacific Coast. In states like Colorado, where a population of 1,000,000 lives on 100,000 square miles, we have a very large number of small isolated high schools, frequently with altogether three or four teachers. Something can be done in the way of consolidation, but distances are too great to accomplish much in this direction. Thus, while in large high schools differentiation, and alternative courses for the two classes of students, is possible, such a program will be a cruel blow to the vast territory between Nebraska and Kansas on the one side and the line of Pacific Coast States on the other.

How is it when we turn to the second plan?

Let us frankly admit that the idea of integrating high school work so that the student takes up each subject at a point where he clearly realizes its importance, may well appeal as an ideal to all of us.

But what shall we find in reality?

In some of our modern high schools, with their thousands of students and hundreds of instructors, we may have organizations sufficiently large and self-contained to carry through such a program. The success of the plan will depend on the presence of a group of teachers of unusual cultural attainment, and fired by an inextinguishable enthusiasm.

I am pessimistic concerning the possibility of making such a plan a success on a nation-wide scale. At the University of Colorado we have had a taste of the difficulties you will encounter. For several years we have been struggling to install an honours course for selected students. Our aim has been to supply upper-class students with a broader cultural basis by surveying and integrating separate fields of knowledge from a broader viewpoint. In our faculty of over two hundred men and women you can count on the fingers of your two hands those in whose ability to supervise such programs you would have implicit confidence. I may say this since I am not one of them. Therefore, much as I realize the danger or prophesying, I feel safe in predicting that any such high school program, applied to the whole nation, or even to a single state, will lead to a vague schematism, under cover of which inefficient teachers will go through motions without accomplishing anything of value. In small high schools, the introduction of any such scheme will usually lead to disastrous results.

What will in many cases happen was foreshadowed at a teachers meeting I attended a dozen years ago. One elderly lady arose and said; "I am a teacher of mathematics, but I do not care whether my students learn mathematics. My object is to make them good American citizens." A second lady arose, a third lady arose, and both applauded this sentiment. The rest sat in silence. And so did I. I did not have the courage to suggest that common decency demanded that, if they were not prepared or willing to perform the duties for which they were paid, it would be a good thing for them to make place for

others who realized their obligations.

Another important question concerning any type of project method is this: What values are sacrificed if we give up the methodical continuous logical development of a branch of mathematics in favour of a piecemeal treatment which takes up a section here, a section there, without any integration? Are we not in some subtle manner losing something of characteristic value and importance?

Thus, of the two methods of approach which are at present in the foreground of interest, each contains elements which are of value, but, if tried indiscriminately and without serious modifications, either seems to an unbiased outside quite unacceptable.

If the mathematics teachers, after careful consideration, arrive at this or at similar conclusions, there can be no compromise. Nothing is in the long run more shortsighted than to sacrifice a principle.

What can we mathematics teachers do in this situation?

There is a large percentage of inefficient, uninspired and uninspiring teachers of mathematics in our colleges and universities; there must be a similarly large percentage of inefficient mathematics teachers in the high schools. (Indeed, it is an open question whether it is at all possible to supply enough efficient mathematics teachers for the schools of this vast country.) But, despite the recognized existence of an inefficient minority in colleges and universities, it is the rule that the instructors of the subject have a decisive vote in prescribing the mathematics curriculum and the contents of the courses and the methods of instruction. Rarely, I think, will the administration of a reputable university risk making a laughing stock of itself by dictating the contents and methods of mathematics courses.

In the high schools, as in the universities, there are thousands upon thousands of well-trained, enthusiastic, inspiring teachers. They are the best judges of what mathematics should be taught in the high schools, how it should be taught, and to whom it should be taught. We have never planned either a defensive or an offensive fight. We have allowed outsiders to dictate to us in our own affairs, and we have lost strong positions. Educationists who are not mathematicians but who have a fair attitude toward mathematics complain that we have no carefully prepared philosophy of our aims, corresponding to, say, Inglis' "Theory of Secondary Education." What we have to offer, seems to be scattered and un-integrated. Our friends desire a readable book in which our whole position is presented in such fashion that it may be discussed and compared with other educational theories!

It is not for me to advise high school teachers; but I may be permitted to express my conviction in these words: Class-room teachers who are thoroughly trained in their subject and all of its backgrounds, who have chosen mathematics teaching in the secondary system because it represents a fulfillment of their deepest desires, who have given years of earnest thought to the problems confronting them —these must feel that it is their right and their duty to share actively in the fight for a reasonable and decent place for mathematics in the high schools.

I do not know what method of procedure may appeal to you. You may prefer to select a group of your own with the whole authority of the National Council back of it, and to stake out your own field of investigation. You may prefer to work through existing committees, such as the joint committee of your Council and the Mathematical Association "On the Place of Mathematics in the Secondary Schools," the progress of which we are watching with profound interest.

Other ways will be open to you if you have the courage of your convictions and insist that you, the actual teachers of mathematics, shall have a decisive voice in all fundamental questions on the curriculum of mathematics in your own schools.

My experience has taught me that administrators are at the bottom of their hearts relieved when they can discuss important problems with a group that speaks with authority, has complete mastery of the facts, and with which they can share responsibility. But as long as the teachers, the real teachers, do not speak for themselves, they must always expect to see others formulate their policies for them.

I started this talk by mentioning that I had some months ago appealed to the Mathematical Association to cooperate with the high school teachers. I end on the same note by pledging the fullest support of the Association in any progressive and aggressive campaign the mathematics teachers of the country may undertake.

The Dandelin Spheres

By LEE EMERSON BOYER
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WITHIN recent years at least four references to the Dandelin spheres have appeared in American mathematical literature.[1] Typical of these references is the first one by Malsch which reads:

> The geometry of the last years (in German secondary schools) should center around the conic sections. . . . The treatment should be a continuation of elementary geometry, beginning with the Dandelin spheres.

and the second one by Lietzmann as follows:

> First there is the stereometric introduction of the section of a circular cone, then the locus definition in a plane with the problem of Apollonius, then the union of both through the contribution of the Dandelin spheres.

The prospective teacher of mathematics ponders the questions: What are the Dandelin spheres? And why are they so called?

As a partial answer to our questions, one should consult Figure 49, page 240 of the *Eighth Yearbook*.[2] A somewhat better picture of the Dandelin spheres with a slightly more meaningful explanation is found in Lietzmann's book.[3] Neither of these sources gives any hint (except that probably Dandelin refers to the name of some person) of why the models are named, or referred to, as they are. Cajori[4] speaks of Germinal Dandelin who contributed a method of approximating the roots of an equation and whose works are found in the Academy of Science, Brussels, Belgium. The *La Grande Encyclopedia* (French) gives a more complete biography of Professor Germinal Pierre Dandelin. (Professor Dandelin was born in Bourget, France, 1794. In 1822 this professor was summoned to the Royal Academy of Brussels and in 1825 he became Professor of Mechanics at the University of Liege. He died in Brussels, 1847). In the thirteenth volume of the La Grande Encyclopedie is found a statement of Dandelin's Theorem which, by translation, reads:

"Sometimes the name, The Theorem of Dandelin, is given to a very old theorem which states that the section of a cone of revolution made by a plane which does not pass through the cone's vertex is an ellipse, parabola or hyperbola. In reality, the Theorem of Dandelin teaches that the foci of the sections are the points of contact of the intersecting plane with the spheres inscribed in the cone and that the directrices of these sections lie in the plane determined by the circles of contact of the spheres with the cones in which they are inscribed."

Casey[5] states an equivalent theorem and attributes it to Dandelin and Quetelet.

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[1] Malsch, Fritz (translated by Ralph Beatley). "The Teaching of Mathematics in Germany since the War." *The Mathematics Teacher.* November 1927. Volume 20, page 366.

Lietzmann, Dr. W. "New Types of Schools in Germany and Their Curricula in Mathematics." *The Mathematics Teacher.* March 1924. page 151.

Lietzmann, Dr. W. "Methods of Teaching Mathematics in Germany." *Fourth Yearbook, National Council of Teachers of Mathematics.* New York Bureau of Publications, Teachers College, Columbia University. 1930. page 48.

Wolf, George. "The Mathematical Collection." *Eighth Yearbook, National Council of Teachers of Mathematics.* New York Bureau of Publications, Teachers College, Columbia University. 1933. page 224 and page 240.

[2] *Eighth Yearbook. National Council of Teachers of Mathematics.* New York Bureau of Publications, Teachers College, Columbia University. 1930.

[3] Lietzmann, Dr. W. *Methodik des Mathematischen Unterrichts, II Teil,* Leipzig. Verlag von Quelle und Meyer. Tafel VII.

[4] Cajori, Florian. *A History of Mathematics.* New York. Macmillan Company. 1926.

[5] Casey, John. *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections.* Dublin. Hodges, Figgis and Company. 1893. pages 365.

Briot and Bouquet[6] present the most comprehensive explanation and specifically attribute the theorem to Dandelin.

[6] Briot, C. and Bouquet, J. C. *Leçons de Geometrie Analytique.* Paris. Librarie Ch. Delagrave 15, Rue Soufflot 15. 1893. page 265.

Cantor[7] also attributes the credit to Dandelin for studying the conic sections by these unique models.

[7] Cantor, Moritz. *Vorlesungen Über Geschichte Der Mathematik IV.* Leipzig. Verlag und Druck von B. G. Teubner, 1924.

How Old Is Ann?

Although it is not at all a difficult problem, it occurred to me that perhaps the readers of *The Mathematics Teacher* might be interested to see the simple algebraic solution to Dr. A. A. Bennett's "How Old Is Ann?" printed at the end of his article in the November (1937) issue.

To arrive at the correct set-up we write the problem as follows:

The combined ages of Mary and Ann are 44 years. Mary is twice as old as Ann was (x years ago) when Mary was half as old as Ann will be (y years hence) when Ann is three times as old as Mary was (z years ago) when Mary was three times as old as Ann. How old is Ann?

Now in addition to the x, y, and z above, let m equal the number of years in Mary's age (now) and a equal the number of years in Ann's age (now).

The following 5 equations in 5 unknowns are easily written:

1 $\quad a+m=44$

2 $\quad m=2(a-x)$

3 $\quad m-x=\frac{1}{2}(a+y)$

4 $\quad a+y=3(m-z)$

5 $\quad m-z=3(a-z)$

Solving gives the following answers:

$a=16\frac{1}{2}$; $m=27\frac{1}{2}$; $x=2\frac{3}{4}$; $y=33$; $z=11$.

So Ann is $16\frac{1}{2}$ years old.

Note that this is based on the fact that the last clause of the long sentence of the problem is understood to mean ". . . when Mary was three times as old as Ann was *then*." If we understand it to mean ". . . as Ann is *now*," we have the same first four equations, but equation (5) is now

$$m-z=3a.$$

This seems possible enough, until we find by solving that x comes out negative, which is not admitted by the wording of the problem. To check the answers, we have but to substitute as follows:

The combined ages of Mary ($27\frac{1}{2}$) and Ann ($16\frac{1}{2}$) are 44 years. Mary is ($27\frac{1}{2}$) twice as old as Ann was ($13\frac{3}{4}$, $2\frac{3}{4}$ years ago) when Mary was ($24\frac{3}{4}$) half as old as Ann will be ($49\frac{1}{2}$) when (33 years hence) Ann is ($49\frac{1}{2}$) 3 times as old as Mary was ($16\frac{1}{2}$) when (11 years ago) Mary was ($16\frac{1}{2}$) 3 times as old as Ann ($5\frac{1}{2}$).

After I had shown this problem, and hinted at its solution, to my second year algebra section, I received the following verse from one of the boys:

With Apologies to the "Purple Cow"

I saw a problem yesterday,
  And now I live to rue it;
For, one thing only can I say:
  I'd rather see than do it!

C. Stanley Ogilvy
Berkshire School
Sheffield, Mass.

The New Arithmetic Course in the Chicago Elementary Schools

By William H. Johnson
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Arithmetic in the early days of American education was a subject relegated to the tradesmen and industrialists, but not considered worthy of the attention of those destined for higher learning. And so for about a century, the schools, in particular those which prepared for college, laid no stress upon the subject. But conditions changed in America. Farmers were no longer the only ones to be considered. Tradesmen and industrial workers were becoming a substantial part of the population; and so arithmetic came into its own. Books on the subject were in the nature of guidebooks in arithmetical computations and solutions covering, as far as possible, all the needs of business. To these a business man could refer to find help in solving any problem with which he was confronted.

When arithmetic textbooks came into use in the schools, solutions were omitted, and thus children were compelled to engage in calculations of a wider variety and scope than any one business man could ever possibly need. Pupils were confronted with computations involved in businesses of which they had no knowledge. Problems dealing with aspects entirely foreign to the children were justified on the ground that they developed the "faculties" of the mind. Drill was often given in such work far in excess of normal adult needs. Children were forced to attempt an impossible load which meant not only failure from the very beginning, but also the lack of mastery of an essential tool. Arithmetic came to be considered a mental discipline.

For some time progressive educators throughout the United States have been advocating an arithmetic which would more adequately meet the needs and interests of the children—an arithmetic with interesting and meaningful situations within their experience and comprehension. Many studies and investigations have centered attention upon the subject, and have shown the desirability of reducing arithmetic to a practical, sane, sensible basis.

Long before he has entered school the child has had many, though limited, experiences with numbers in his everyday normal activities. He has counted members of his family, his toys, his age; he has some conception of space through words as "top," "bottom," "up," "down"; he has some idea of time, as "today," "tomorrow," "yesterday," "time to get up," "time to go to bed"; he has experienced measures, quantity, size, groups, and money.

It is upon these beginnings that certain departures from the traditional courses of study in arithmetic are made. These departures will be better understood if some disconcerting facts in most arithmetic teaching are taken into account:

1. More children fail in arithmetic than in any other subject.
2. Many school children dislike arithmetic, and many adults think it a dry and uninteresting subject, a necessary evil.
3. There is a too-prevalent habit among children of counting on their fingers when adding.
4. Poor results in the most elementary processes of arithmetic are registered in tests given at the end of the sixth grade.
5. Children commonly have perverted ideas with respect to mathematics, and lack adequate mathematical preparation for entering high school.
6. Many children and adults are vic-

tims of an arithmetic inferiority complex because they have been taught abstract processes before they were old enough to comprehend them.

7. The business world complains that products of our elementary schools are inaccurate in the fundamentals of arithmetic.
8. We now have quite conclusive evidence from certain carefully carried out experiments enabling us to decide in what grades the important parts of arithmetic can best be taken up, and what are the best methods of teaching to employ.
9. There are studies in social and business usage now available indicating just how much arithmetic pupils really need.
10. Studies in maturation and growth in children help us better to understand their limitations at certain ages—what they can do, and what they can not do.

In view of the above arguments we of Chicago have inaugurated many departures from the customary traditional procedures in the teaching of arithmetic in the elementary school:

(1) All formal arithmetic requirements have been taken from grades one and two. There are no definite drills for memorization and mastery of the addition and subtraction combinations, in order to attain automatic recall, definite goals, or definite time limits. This is because there is an abundance of evidence now available, which shows that in the early stages the learning of number (which is an abstraction) is a slow process filled with many, varied, and intricate mental procedures, such as these—

A. A child has to learn the cardinal number names, as "one," "two," "three," and their proper sequence.
B. He must learn the ordinal number names, as "third," "seventh," etc.
C. He must then learn to discriminate in meaning between these and the cardinal numbers (i.e., between third and three, etc).
D. He must learn the location of each number in the series; for example, that seven comes after six.
E. He must learn that the last number in a series is also the group number of the whole series.
F. He must write legibly the number digits.
G. He must learn simple fractional parts of wholes.

When the above concepts and procedures have been woven into experiences that come naturally into the life of the average child, he is ready for formal addition and subtraction.

(2) The difficult aspect of each arithmetic topic has been moved up to a grade placement as far as distribution in grades allowed. The topics requiring most promotion are division and common fractions.

(3) A third departure is a recommendation in procedures that have been moot questions. Where now there is sufficient experimental evidence for the superiority of one method over another, the superior method will be recommended. Such changed methods apply to the following topics—

A. The first teaching of addition combinations.
B. The teaching of subtraction where the minuend figure is smaller than the subtrahend figure.
C. The teaching of division with a one-place divisor.
D. The teaching of the estimating of the quotient figures in long division.
E. The teaching of the placing of the decimal point in the quotient in division of decimals.

So that principals and teachers may keep abreast with the most modern methods of presenting and teaching arithmetic, demonstration centers have been established. These arithmetic demonstration centers, in charge of an expert from the Chicago Normal College, are as follows:

Barton Elementary School, Bryant Elementary School, Bryn Mawr Elementary School, Douglas Elementary School, Fiske Elementary School, Lloyd Elementary School, Palmer Elementary School, Skinner Elementary School, Trumbull Elementary school.

The theme in this course, if a course of study may have a theme, is *Teaching Arithmetic with Meanings*. The committee, which was composed of Chicago teachers who are experts in the field and who worked in consultation with leading authorities from universities, feel that advancing topics to a place where pupils are more mature will make arithmetic a more meaningful and more interesting subject, and at the same time obviate much of the drill for maintenance of skills that have no immediate use.

Parents, pupils, and teachers should welcome this timely change. Parents will not be annoyed by questions in arithmetic homework. Pupils will grasp the first presentation of a topic more readily, will understand it better, and will retain it longer. Teachers will feel that their teaching has not been in vain and will consequently experience a new joy in their work.

THE ART OF TEACHING

Examinations for Analysis

By WALLACE B. GRAHAM
Garden City High School, Garden City, New York

THE practice of giving a test at the end of each teaching unit or series of units is an accepted method of instruction. A lengthy justification of this procedure is not my aim in this short article. To achieve some sort of an objective measure of the pupil's ability is probably the most apparent purpose of a test. Whether this rating is of benefit to pupil or teacher is of little importance. To one who has been handling a class group regularly, both the general intelligence and specific subject abilities of each member of the class, are quite evident. A justification of the given mark in the mind of the pupil seems to me to be of definite importance. The test tends to do this, for it minimizes the subjective ranking, thus creating a better understanding between pupil and instructor.

However, I wish to present another purpose of the examination, one that is too often overlooked by teachers,—an examination for the purpose of "analysis of teaching." To give a test with this aim in mind, involves additional time and care, both in preparation and in analysis, but the results amply repay the teacher for the extra work.

An analysis of teaching is of little practical value to the class at the end of the course, but is absolutely necessary during the teaching of a course. Several weeks before the end of each quarter or semester is a logical time for such an examination. It will then give both pupil and teacher an opportunity to attack the weak spots, and so cut down on the possible failures at a time when it becomes harder to correct difficulties or omissions.

We teachers are not infallible. We are liable to make mistakes, both of omission and commission. The assumption of knowledge on the part of a student is one of the most dangerous of all errors. The illustrative example given in this article brings out this point, for the writer found upon analysis that he had neglected to give enough drill in the problems involving the order of operations in algebra. Pupils as a rule tend to hide their lack of knowledge, rather than express it. Many will go through an entire course without asking a question. Because of the fact that one-third of the class asks questions, the teacher tends to overlook the other two-thirds, and assumes that all difficulties have been overcome.

In the usual test or examination of twenty or thirty problems, there is generally a hodge-podge of questions that measure the pupil's knowledge of the subject. The careful teacher obeyed certain rules such as the arrangement of problems in order of difficulty. He may have attempted to omit "catch" questions. He may have tried to use a suitable vocabulary. He has probably tried to time the test correctly. But, in most examinations, there is almost no opportunity for the teacher to analyze the paper and find to what extent the pupil has mastered individual processes.

Examinations for analysis differ according to the amount of subject matter to be analyzed. The specific weaknesses of a group who have just studied simple equations may be analyzed by giving groups of problems, each group selected to measure a single type of equation. If the

test is to be given to analyze the teaching of a whole semester, it will cover a series of subjects, such as the fundamental operations, equations, etc. The main objective, however, is that the results must show the teacher and pupil just where the difficulty has been. It is not a question of "Did he fail?", but, "Why did he fail?"

The following illustration with the tabulated results is given to illustrate the procedure. The test under consideration was prepared for four classes in algebra, and was given several weeks before the end of the first quarter of the year. The test was made up in ten parts, covering the subject matter as follows:

1. Writing with algebraic symbols.
2. Solution of area formulas. Formulas not given, but previously assigned.
3. Solution of mixed formulas. Formulas given.
4. Solution of volume formulas. Formulas given.
5. Order of Operations.
6. Substitution—easy.
7. Substitution—medium
8. Substitution—hard.
9. Simple addition in algebra.
10. Construction and Solution of Formulas.

Most of the sections had about five problems. Several had ten. Each was arranged in order of difficulty. No attempt was made to check on arithmetic abilities or skills. Numbers were used that would not cause trouble. To use complicated fractions and decimals would have confused the results in analyzing the algebraic difficulties.

One hundred seven pupils took the test. Both the median and the average was 65. The mode was 58 and was used, in this particular case, as the "so-called passing mark" for this test.

A few of the conclusions resulting from this test are given below. Some of these were already evident to the teacher. Others were both startling and surprising. On the whole, the test was decidedly worth while. It improved the morale of the classes in as much as most of them became interested in the analysis instead of the

The results, by classes, were as follows:

1st period—Average 64 median 64
2nd period—Average 66 median 69
3rd period—Average 68 median 69
4th period—Average 60 median 58

The results, by sections of the test, were as follows:

Section No.	1st period	2nd period	3rd period	4th period
1.	74%	75%	80%	66%
2.	71%	60%	74%	60%
3.	88%	82%	86%	80%
4.	90%	92%	93%	90%
5.	69%	58%	78%	60%
6.	89%	93%	88%	83%
7.	71%	81%	76%	70%
8.	33%	46%	30%	42%
9.	43%	58%	51%	40%
10.	16%	23%	28%	13%

The results, by pupil, (in part) were as follows:

Section—Pupil—	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1.	10	10	10	8	6	6	8	8	8	10	10	8	6	8	8	4	8
2.	6	10	8	6	8	8	8	6	10	8	6	4	10	6	8	10	8
3.	10	10	10	10	10	10	8	8	10	10	8	10	8	10	6	10	10
4.	10	10	10	8	10	10	10	10	10	8	8	10	8	10	8	10	10
5.	10	10	8	10	8	10	10	4	6	8	4	8	10	10	8	8	6
6.	10	10	10	10	10	10	10	10	10	10	10	10	10	8	10	6	10
7.	10	8	10	10	10	6	8	8	8	6	8	6	6	6	8	8	10
8.	10	5	10	10	5	0	5	10	5	5	0	5	0	5	5	5	0
9.	8	8	8	8	10	6	7	8	4	4	6	6	8	2	4	2	0
10.	8	8	3	0	0	9	0	0	0	0	8	0	0	0	0	0	0
Total mark—	92	89	87	80	77	75	75	72	71	69	68	67	66	65	65	63	62

mark. In fact, the entire discussion in the different classes was on the basis of the analysis. Several afternoons were devoted to remedial work, based on the results of the test.

1. All four classes showed poor results on the subject of *Order of Operations*, giving evidence of insufficient teaching and drill on the subject.
2. The best class (3rd period) showed the lowest average on Section 8, which were the hard problems in *Substitution*. This result was, of course, a complete surprise. The only possible solution is that the teacher assumed knowledge on the part of this class, and therefore did not give enough aid to the weaker half of the group.
3. An analysis of all of the 107 papers is not possible in this limited space. Let us glance at the results of Pupil H as given above. He showed two decided weaknesses,—a lack of understanding concerning the order of operations and an inability to construct a formula from a word statement. He had also neglected to learn the formulas assigned, but showed ability to solve a problem if the formula was given. One mistake showed a misunderstanding of the use of 0 as a multiplier, another of the use of 0 as an addend.

The President's Report, 1936-1938

During my term of office the National Council has scheduled five meetings as follows:

1. Reed College, Portland, Oregon, June 27–29, 1936
2. Chicago, Illinois, February 19–20, 1937
3. Detroit, Michigan, June 28–30, 1937
4. Indianapolis, Indiana, December 29, 1937
5. Atlantic City, New Jersey, February 25–26, 1938

Summer Meeting

It has been the policy of the Council to continue the summer meetings at the time of the N. E. A. begun in 1935. There has been a steady and healthy increase in attendance each year, frequently on the part of the classroom teachers unable to attend our annual meetings. The Department of Secondary Education of the N. E. A. of which George R. Rankin, Boys' Technical High School, Milwaukee, Wisconsin, is president, has lent fine cooperation. The topic for our joint meeting in 1936 was "Trends in Mathematics Instruction in High School and Possible Use of New Curriculum Materials," in 1937, "Function of Extra Curricular Activities in Mathematics." In a recent letter from Mr. Rankin he suggested that the topic for the June, 1938 convention in New York should be "Readjustments in Teaching High School Mathematics to Meet the Needs of Brilliant, Non-academic and Other Groups," or "Meeting New Needs and Group Needs in High School Mathematics."

Holiday Meetings

Because the A. A. A. S. met at Atlantic City and the American Mathematical Society and the Mathematical Association of America met at Duke University, it seemed wise to omit the meeting in December, 1936. In December, 1937, a one-day meeting was held at Indianapolis, Indiana, one-day because the hotel space in Indianapolis was so very limited. Our local chairman, Walter Carnahan, was fortunate enough to secure for our meetings the Central Shrine of the World War Memorial Building, one of the most beautiful auditoriums in Indiana. This auditorium seated only 500, but even though the Indiana committees sent out 700 letters and your president sent out 900, the auditorium was not full. More people than were expected attended the luncheon at which Mr. Reeve spoke about the work of the Joint Commission of the *National Council of Teachers of Mathematics* and the *Mathematical Association of America* on "The Place of Mathematics in Secondary Education." It was necessary to open an extra room for the overflow.

Annual Meetings

Because the American Association of School Administrators met in New Orleans in February, 1937, the Board decided to meet in Chicago. A satisfactory hotel with ample room for meetings and good service was easily secured. The local clubs cooperated in a very fine manner to make this meeting a very successful one. A fine exhibit was organized and placed in the foyer. It included some classwork of pupils, teacher and pupil made helps in teaching, an exhibit of commercial instruments, a collection of pictures suitable for mathematics classrooms, a library of high school texts, copies and lists of books suitable for high school libraries and teachers' libraries and the beginning of a fine collection of posters contributed by the high schools of Chicago and vicinity.

Owing to the great demand for hotel space, it was much more difficult to arrange for the Atlantic City meeting. The teachers of mathematics of the state of New Jersey were our hosts. They, under the leadership of Professor Mallory of our Board, made general arrangements for hospitality and exhibits. Mr. Lidell of

Atlantic City, our local chairman, worked hard to secure a good hotel, but it was difficult and expensive for our members to reserve rooms at the headquarters hotel because other organizations also reserved rooms for these days as well as the following week. Moreover, definite rooms for our conferences could not be assigned to us until late in the fall because other departments had first choice.

Relations with N. E. A.

Through the efforts of H. C. Christofferson we have been recognized as an allied organization of the N. E. A. and it has been a little easier to plan summer meetings and have our programs inserted in the general program of the N. E. A.

Relations with Other Mathematical Organizations

Because our problems are not only our own but, in a measure, also concern those who are teaching mathematics in colleges and universities, the holiday meetings have helped greatly in bringing about a better understanding between these groups. The work of our Joint Commission is also of great help in fostering a better relationship and understanding between our organizations. At Indianapolis, the president of the Association, Dr. Kempner, kindly spoke to our group (see page 117 of this issue) and at a dinner your president was the guest of the combined group, a kindly recognition of our high school teachers. Our former president, Mr. Betz, has been elected a member of the Board of the Association. It would be well if more of their membership belonged to the Council and more of us were members of the Association.

Dr. Slaught

During the last days of May, we buried our honorary president, Dr. Slaught. Fortunately, the last flowers sent to his funeral were not the only ones we were privileged to bring him. At our Chicago meeting, it was possible to send him flowers to brighten his room and a little booklet of resolutions to express our kind feelings toward him. Those who were near and dear to him tell us that these tributes did much to brighten the last days of his life. Although he was unable to leave his room and be with us, the Allied Radio Corporation made a record for us of his address to the Council and Dr. Reeve and Mrs. Logsdon came before the group to tell of their association with Dr. Slaught as students, fellow workers and fellow teachers. For the booklet of resolutions we are indebted to two Proviso teachers, Miss Wilhelmina Marm, who took care of the lettering and art work and Miss Florence I. Otis, who secured the leather covers for us.

Committees

The Council is at present sponsoring the work of two committees, that of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics on the Place of Mathematics in the Secondary Schools, and the committee on Arithmetic. It is hoped that the results of the hard work of both of these committees will be made available to all teachers of mathematics, perhaps in future year books. It has been the aim of the president to help the work of these committees by giving them a part in the programs of our meetings, thus enabling them to present their cause and enlist the active interest of our teachers.

Innovations

This has seemed a good time for the Council to expand, to widen its interests. For this reason two sections were scheduled for the same hour at Chicago, Detroit, and Atlantic City. This change has met with a degree of success since it enables more people to speak on a variety of topics. It also makes the work of the president more difficult. I would like to suggest that committees (not too large) be appointed to study various questions of interest to teachers of mathematics, that these committees work, keeping the meet-

ings of the Council in mind and that they recommend to the president topics which should be discussed at these meetings and likely speakers for these topics. In this way our meetings would have more in them which is new and helpful, not only in keeping our teachers of mathematics abreast of the times and more capable, but also in planning this progress. I would suggest committees with the following interests: junior high school work, visual and sound education, enrichment of curriculum, arithmetic, geometry, classroom planning, exhibits, publicity of the work of mathematics, individual differences as they concern the bright pupil and the non-college going pupil, etc., teacher training, suggested programs for affiliated clubs and many others.

The Discussion Luncheon was first introduced at the Chicago meeting. For these, table leaders have been permitted to choose their own topics. These luncheon meetings have met a very definite need in that they give the members, not only a chance to listen, but also to express themselves. They are also full of helpful suggestions for future programs in that they show a trend as to the kind of topics which interest teachers and they decidedly promote friendliness and professional acquaintance, which our people enjoy.

At three of our meetings exhibits helpful to high school teachers of mathematics have been or will be attempted. The one in Chicago has already been described. In Indianapolis the exhibit was not as large but had many interesting features among them a fine display of games based on mathematics. At Atlantic City the teachers of the State of New Jersey sponsored the exhibit. This part of the program has also been found worth while. It helps those who attend our meetings to get new ideas as to how the work of their department can be brought before the public. It helps people get acquainted, it is a valuable experience for those who sponsor the exhibit, and it gives more people an opportunity to take an active part in our meetings.

Membership and Income

There has been a steady increase in membership during the last two years. You will hear more of this in other reports for much of this is due to the very fine work carried on by Florence Brooks Miller and Mary Kelly in their organization of state representatives and affiliated groups. The friendly dinners at annual meetings have been abandoned in favor of serious talks on problems in different sections of the country. This work with state representatives and affiliated groups requires more money than is apportioned to it, but if faithfully carried on it must, in time, bring with it the cooperation of a much greater percentage of the teachers of mathematics in the United States. I would like to suggest that the Board or some committee named for this purpose make themselves available at our various meetings, in the front or back of the auditoriums for purposes of consultations on, not only how these affiliated groups for local improvement may be organized, but how they carry on their meetings and activities to a greater degree of satisfaction and mutual helpfulness. At Indianapolis, I was approached by several teachers for just such information when I was too busy to give them as much attention as they needed. More and more, it is necessary that our National Council of Teachers of Mathematics be truly national and important, not only to our best, but to all of our teachers of mathematics.

During my term, my office has sent out about 3,000 letters telling of the work of the Council and its meetings, mostly to teachers who are not members of our organization. This has been done in addition to the work carried on by Mrs. Miller and Miss Kelly and their representatives and chiefly to promote a greater attendance at our meetings.

The President's Desk

As the Council grows and its activities multiply, the work which passes over the president's desk is steadily increasing.

Leaving entirely out of consideration the talks a president is expected to make, with or without financial assistance, it is impossible for one person to know enough about all of these activities to care for them in a satisfactory manner. The committees previously mentioned would be a great help in planning successful meetings. The Board must begin to think about a more central organization which will promote the activities of the Council, increase the membership, advise and assist affiliated organizations, and many more things too numerous to mention here.

Acknowledgments

I am deeply indebted to the very able local chairman: Lesta Hoel of Portland, Oregon; Elsie and J. T. Johnson of Chicago; Hildegarde Beck of Detroit; Walter Carnahan of Indianapolis; and Burton Lidell of Atlantic City for the splendid work they have done with their committees in making our fine meetings possible and well attended. I am indebted to all members of the Board for their willingness to participate in programs, their helpful suggestions and their support, but most particularly to W. D. Reeve, editor-in-chief of *The Mathematics Teacher* for his fine and prompt cooperation, past presidents Hassler and Betz, and Secretary Schreiber for their help and encouragement, to E. A. Bond, C. L. Thiele, H. C. Christofferson, Mary Potter, Vera Sanford, Florence B. Miller, C. B. Upton and many others for their help in program planning, to W. S. Schlauch and H. C. Christofferson for their work on income and expenditures and to all my many friends among the teachers of mathematics who have wanted the work of the Council to prosper and glow.

Respectfully submitted,

Martha Hildebrandt, *President*

◆ EDITORIALS ◆

The National Council's Retiring President

Miss Martha Hildebrandt is retiring from the presidency of *The National Council of Teachers of Mathematics* this month after two years of devoted and notable service. Under her leadership the Council has grown not only in numbers but in prestige and influence among school people generally. Anyone who reads her report on the preceding pages of this issue will obtain some idea of the time and energy she has devoted to the cause of mathematics, but only those who have worked closely with her in the organization will know how much she has given of her real self and how unselfishly she gave it. She has been a credit not only to the Council but also to her school where she is a teacher of secondary mathematics. The best that *The Mathematics Teacher* can do is to wish Miss Hildebrandt a long and rich life wherever she may be. We trust, however, that the National Council may continue to receive her wise advice and that she may have the rare pleasure of seeing us ultimately reach our goal of ten thousand members before many years have passed.

W.D.R.

Orpha Worden

The Mathematics Teacher is grieved to announce the death in Miami, Florida, recently of Miss Orpha Worden, formerly Associate Professor of Mathematics at Wayne University in Detroit, Michigan, following a fall which she sustained in a dimly lighted corridor. Miss Worden retired from teaching on February 1, 1937, after forty years of devoted service. She has long been an active and interested member of the National Council of Teachers of Mathematics and her death will be a distinct loss to our cause. Interment was at Grand Ledge, Michigan, on February 12th.

W.D.R.

State Representatives of the Council

The work that Mrs. Florence Brooks Miller has been doing through the various state representatives of the National Council has been of great help to *The Mathematics Teacher* in its campaign to obtain 10,000 members. However, a great deal of work needs to be done before we can reach that goal and particularly before many teachers of mathematics can be made to realize their responsibility to the cause they represent. Letters which come in almost daily to the office of *The Mathematics Teacher* indicate a lack of genuine interest in our cause which is almost appalling. If possible, state representatives should know all teachers of mathematics who should be members of the Council and should then use every effort to get them to become members of the Council. *The Mathematics Teacher* is ready at all times to assist by sending the mailing list of the various states to the proper persons.

W.D.R.

IN OTHER PERIODICALS

By Nathan Lazar
Alexander Hamilton High School, Brooklyn, New York

1. Ironmonger, E. E. "A Method of Teaching." *The Mathematical Gazette.* 21: 271–273. October, 1937.

In 1816, Dr. John Taylor published an English translation of a treatise on Hindu arithmetic and geometry, written by Bhascara Acharya (1114–c. 1185). In the appendix, the translator included "A Short Account of the Present Mode of Teaching Arithmetic in Hindu Schools." In the article under review a description is given of the contents of the appendix. The religious ritual that students had to perform on joining the school is related in great detail. The syllabus followed in the school, the method of instruction, and the arithmetical tables memorized are all clearly described.

2. Jelitai, Jósef. "The History of Mathematics in Hungary before 1830." *National Mathematics Magazine.* 12: 125–130. December, 1937.

A brief sketch of Hungarian mathematicians between the years 1499–1830. In the latter year appeared the first mathematical work of Wolfgang Bolyai, who constitutes "with his son John, the greatest double star of the mathematical firmament in Hungary."

3. Kasner, Edward. "New Names in Mathematics." *Scripta Mathematica.* 5: 5–14. January, 1938.

A stenographic report of a lecture delivered before the Forum of the Society of Friends of Scripta Mathematica. The title is misleading, for the article does not deal with brilliant young mathematicians but with new words or terms that have found their way into more or less recent mathematical literature.

Many ordinary words such as the following are used in mathematics in a peculiar sense: group, family, ring, simple curve, function, domain, integration, differentiation, and transcendental. The following examples are of mathematical terms coined by Prof. Kasner himself: turbine, polygenic functions, parhexagon, hyper-radical or ultra-radical, googol and googolplex. A googol is defined as 10^{100}. A googolplex is 10^{googol}, which is $10^{10^{100}}$.

The above bare summary is in no way indicative of the keen mathematical insight, the sly wit, and refreshing humor with which the article is filled.

4. Loria, Gino. "Teaching the History of Mathematics." *The Mathematical Gazette.* 21: 274–275. October, 1937.

Before introducing historical concepts into the mathematical curriculum of the secondary schools, the future teachers must be prepared for that step. Professor Loria relates the experiences that he had with a course designed for teachers, and gives a brief outline of the topics that should be included in such a course.

5. Moore, Charles N. "Mathematics and Science." *School Science and Mathematics.* 38: 41–52. January, 1938.

An interesting study of the interrelations between pure mathematics and experimental science. The author concludes by insisting on "the essential unity of all that can be designated as science. One outstanding purpose lies at the basis of all scientific endeavor, as the etymological origin of the word science indicates. We wish to increase our knowledge, both of ourselves and of the world about us. In carrying out this purpose each individual works best along the lines dictated by his own tastes and inherent capacities. Some of us are for this reason mathematicians, and more particularly mathematicians working in certain special fields. If the theories we develop had no bearing at all on other scientific work, they would still have a value as exhibiting the capacities of the human mind. But the interrelation of the various scientific fields adds much to the solidarity of scientific interests. We should, therefore, rejoice that the relationship between mathematics and the other sciences are of such great service in the general development of scientific thought. It may not make our work any more interesting to ourselves, but it adds much to its broad human interest."

6. Nyberg, Joseph A. "New Uses for the Decimal Point." *School Science and Mathematics.* 38: 59. January, 1938.

The financial pages of daily newspapers use the decimal point to indicate thirty-seconds of a dollar. Thus when certain U. S. Treasury Notes are quoted as selling for 106,25 it does

not mean for 106 dollars and 25 cents, but for 106 dollars and 25 thirty-seconds of a dollar.

"Psychologists have also found a new use for the decimal point. Rather than write 12 yrs. 5 mo. as a pupil's age (chronological, mental, or achievement in some activity) we now frequently see this written as 12.5. The decimal point thus represents twelfths instead of tenths.

"So far the new uses have not found their way into textbooks on arithmetic, but sooner or later the schools will need to teach what the rest of the world is using."

7. Pidduck, F. B. "The Reform of Mathematics." *The Mathematical Gazette.* 21: 249–255. October, 1937.

Many refreshing and shrewd comments on various problems of interest to the mathematician and the teacher of mathematics.

"If the history of mathematics has any lesson to teach, it is surely that the extreme development of a branch containing only a limited stock of ideas is an unhealthy growth, and that progress has been made at intervals by infusion of new ideas, treated at first with utmost simplicity. . . . The properties of integral numbers are no more worthy of attention than a common puzzle. If God created the integers, He must have had other help in the manufacture of those who think of nothing else. . . . The principal pitfall in applied mathematics is, of course, the temptation to turn it into pure mathematics . . . The more the study of physics reveals new fields, the less need there is to cultivate the old ones intensively. . . . A well designed course of geometry should have the elements written as far as possible in current prose: nothing to be proved that an intelligent and wholly ignorant reader would not regard as obvious. . . . The arch enemy is the examiner who seeks to narrow the schedule and raise the standard of technical achievement within it. If the position assumed in this essay is a sound one, the proper solution is found in the opposite direction. Most searching questions can be put without a trace of artifice in the mathematical theory of electricity. But whatever the object, let us have a wide range and a liberal choice. This alone can save us from a dangerous and ever present menace."

8. Reeve, William D. "The Place of Mathematics in Modern Education." *Scripta Mathematica.* 5: 23–31. January, 1938.

A summary of this article will be given in this department when the concluding part appears in a subsequent issue of *Scripta Mathematica.*

9. Russell, David W. "Introducing Mathematical Concepts in the Junior High School." *School Science and Mathematics* 38: 6–19. January, 1938.

Mathematical concepts may be introduced to students on the junior high school level by means of photographs of various structures found in the city and in the country. Mathematical operations can be learned in connection with an insurance company organized in the class room for "protection" against loss of pencils, books, etc.

Despite this human approach to mathematics the importance of the development of skill and habits of accuracy is not overlooked.

10. Sanders, S. T. "A Phase of Mathematical Research." *National Mathematics Magazine.* 12: 110. December, 1937.

In 1916, G. A. Miller said that "it appears likely that the twentieth will produce, as the nineteenth century has done, much more new mathematical literature than the total existing mathematical literature at its beginning." Playing with the above idea, the writer of the article wonders whether the present rapidly increasing volume of mathematical output will "finally mark a time when it shall be impossible for one to know with certainty whether his product is new, or is a mere duplication of another's previously published or unpublished discovery."

11. Short, W. T. "Hyperbolic Solution of the Cubic Equation." *National Mathematics Magazine.* 12: 111–114. December, 1937.

A description of a method, not generally known, of using hyperbolic functions in the solution of the cubic equation.

12. Smith, D. E. "Algebra of 4000 Years Ago." (Second Note.) *Scripta Mathematica.* 5: 15–16. January, 1938.

A brief report of the contents of Vol. III, part 3, of *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik,* by O. Neugebauer. The first note on Parts I and II appeared in *Scripta Mathematica,* 4: 111–125, April, 1936.

13. Speiser, Andreas. "Report of the Euler Commission." *National Mathematics Magazine.* 12: 122–124. December, 1937.

A brief report on the recently published volumes of the Collected Works of Leonhard Euler. It is important "to emphasize, above all, that the Eulerian works have not only an historical interest, but that today they still form a rich store-house for researches."

NEWS NOTES

The annual open luncheon meeting sponsored jointly by the Association of Mathematics Chairmen and the Association of Teachers of Mathematics of New York City will be held on Saturday, March 12, 1938, at 12:30 P.M., at the Essex House, 160 Central Park South, New York City.

The principal speaker will be Dr. Harlow Shapley, Director of the Harvard College Observatory. His topic, "The Current Explorations of Atoms and Galaxies," will be illustrated with lantern slides.

Tickets ($1.75 including gratuities) are obtainable by addressing Mr. Samuel H. Barkan, Benjamin Franklin High School, 309 East 108th St., New York City.

At *Teachers College, Columbia University*, from July 5th to August 13th these will be offered by Professor W. D. Reeve: Mathematics education field course in Germany and England, sailing from New York June 30 with return sailing from Southampton August 19. By Professor C. B. Upton: Reorganization of senior high school mathematics; Teaching arithmetic in primary grades; Teaching arithmetic in intermediate grades. By Professor J. R. Clark: Teaching geometry in secondary schools; Teaching algebra in junior high schools. Professor C. N. Shuster: Modern business arithmetic; Field work in mathematics. By Dr. J. A. Swenson: Professionalized subject matter in senior high school mathematics; Demonstration class in eleventh year mathematics. By Mr. R. R. Smith: Teaching algebra in secondary schools; Professionalized subject matter in intermediate algebra. By Dr. N. Lazar: Demonstration class in demonstrative geometry. By Dr. A. D. Bradley: Navigation and map projections; Professionalized subject matter in junior high school mathematics.

Report of the Work of the Affiliated Organizations of the National Council of Teachers of Mathematics January, 1938

At the St. Louis meeting, 1936, the work with the Affiliated Clubs was assigned to the Second Vice President.

It was stated that "Any local club of Mathematics Teachers may become affiliated with the National Council upon passing a resolution expressing a desire to affiliate and filling out the blank of affiliation supplied by the Secretary. The Certificate of Affiliation is then issued to the club by the Secretary of the National Council. There is no fee for affiliation."

At the time of the St. Louis meeting there were twenty-three affiliated branches of the National Council.

The number reported at the annual meeting in Chicago was thirty-five. The clubs that affiliated in 1936–37 were as follows:

1. *Alabama*—The Mathematics Section of the Alabama Educational Association.

2. *California*—A new branch was organized in this state. At this time the two sections are:

(1) The East Bay Section.

(2) The West Bay Section.

Each section has its own officers. The advisory chairman for both sections is Mrs. Bernice Cochran Johnson, Oakland, California.

3. *Colorado*—This state added a second affiliated branch through the efforts of Mr. H. W. Charlesworth and the late L. Denzil Keigley of Denver.

Mr. Keigley was untiring in his efforts to build up the Sectional meetings in his state. He will be greatly missed at the Colorado State meetings.

The new branch is known as the Southern Division, Mathematics Section of the Colorado Educational Association.

4. *Kansas*—Added a new local club which affiliated with the National Council, the Wichita Mathematics Association. The State Association of Mathematics Teachers affiliated in 1935.

5. *Minnesota*—The Range Mathematics Club was affiliated in 1936. This club has been very active since its organization.

6. *New York*—Three new clubs affiliated with the National Council from this state.

The David Eugene Smith Club of New York City.

7. The Mathematics section of the Western Zone of New York State Teachers Association.

8. The Tri-County Mathematics Section includes Allegheny, Steuben, Livingston counties.

9. *Oklahoma* Council of Teachers of Mathematics was reported for 1936. It has later developed that it is another name for the Mathematics section of Oklahoma Educational Association affiliated in 1928.

10. *Georgia*—The Mathematics Section of the State Educational Association was affiliated in January of 1937, just before the Chicago meeting.

At the annual meeting in Chicago the second Vice President became the first Vice President. Miss Hildebrandt, President of the National

Council asked that the work with the Affiliated Clubs be continued by the first Vice President.

This year several Associations of Teachers of Mathematics have affiliated.

1. The Nebraska Section of the National Council of Teachers affiliated early in the year at Lincoln, Nebraska. This is one of very progressive affiliated clubs. Miss Amanda Johnson, the local Editor has brought this club forward rapidly.

2. Through the efforts of Miss Kate Bell and her associates in Spokane, we have added a new branch of the Council in the Northwest: The Inland Empire Council of Teachers of Mathematics.

Tennessee—With the assistance of President Martha Hildebrandt, two new branches of the National Council have been added in this state.

3. The Tennessee Valley Mathematics Association.

4. The Mathematics section of Middle Tennessee Educational Association.

5. *Iowa*—Due to the work of Miss Ruth Lane, University High School, Iowa City, the Iowa Association of Teachers of Mathematics is completing its affiliation with the National Council.

President, Mr. Vernon Price.

Secretary, C. W. Wester, Iowa State Teachers College is active in this work.

6. *Missouri*—At one time Missouri had an affiliated club at Springfield. This club seems to have disbanded. No trace of it is found today. Through the work of G. H. Jamison of Kirksville State Teachers College, the Mathematics section of the Missouri State Teachers Association is affiliating with the National Council. R. E. White of North-East High School, Kansas City, Missouri is president of this association.

The Council now has the following affiliated branches. The number in each state is indicated: 1—Alabama; 2—California; 2—Colorado; 1—Georgia; 3—Illinois; 1—Iowa; 2—Kansas; 1—Kentucky, 1—Maryland; 1—Michigan; 2—Minnesota; 1—Missouri; 2—Nebraska; 6—New York; 1—North Carolina; 1—New Mexico; 2—Ohio; 3—Oklahoma; 1—Pennsylvania; 1—South Carolina; 2—Tennessee; 1—West Virginia; 1—Louisiana-Mississippi; 1—Washington-Idaho.

The names of the Affiliated Clubs in each state with their officers follow:

Letters have been received from the President of eighteen or twenty of these clubs within the past month. Several are sending delegates to the Atlantic City meeting.

It is hoped that all of these affiliated clubs are active in their state or local community. Great advancement will be made in the teaching of Mathematics and closer cooperation among the teachers of mathematics will follow if the teachers of Mathematics in each state will organize and affiliate with the National Council.

The usefulness of the state and local branches of the National Council has been shown at our annual meetings in the past few years.

The Chicago Men's Mathematics Club and the Women's Mathematics Club of Chicago did a wonderful piece of work at the time of the annual meeting in Chicago, February, 1937.

The Detroit Mathematics Club had everything well planned for the entertainment of visiting Teachers of Mathematics at the summer meeting.

The clubs in the east are making great preparation for the annual meeting in Atlantic City in February, 1938.

The Round-table meetings held in connection with the annual meetings of the National Council for State Representatives and Delegates from the various affiliated organizations will be a means of arousing interest and extending this work. Mrs. F. Brooks Miller of Cleveland will have charge of the Round-table at Atlantic City.

There are great opportunities for constructive work for the affiliated clubs.

Mary Kelly
Vice President of the National Council

Note—The complete list of affiliated clubs with their officers was submitted by Miss Kelly but is too long to include in this duplicated report.

A Joint meeting of the Range Mathematics and Science Clubs was held at the Fay Hotel in Virginia, Minn. on Thursday December 2, 1937.

The program consisted of:

Songs by the Brigadiers' Quartet (Virginia Junior College).

Address, "The Science and Mathematics Teachers Look at Consumer Education," by Mr. Walter A. Kumpf, Virginia Junior College.

Address, "Scientific Methods Used by the Department of Conservation," by Dr. A. G. Horn, Biologist of the Superior National Forest.

Miss Evelyn O'Neil, Virginia Junior High School, and Mr. E. D. Sayles, Virginia Junior College, President of the Science Club, were in charge of the arrangements.

H. G. Tiedeman, *Chairman*
Mt. Iron, Minn.

The sixty-first regular meeting of the Association of Mathematics Teachers of New Jersey

was held in Convention Hall at Atlantic City on November 13, 1937. The program follows:

"The Value of Certain Logical Concepts and Principles in the Teaching of Geometry," Mr. Nathan Lazar, Alexander Hamilton High School, Brooklyn, N. Y.

"Report of the Special Committee on Publications," Professor Albert E. Meder, Jr., New Jersey College for Women.

"Plans for the Spring Meeting of the National Council of Teachers of Mathematics," Professor Virgil S. Mallory, Montclair State Teachers College.

Officers of the Society

Mr. E. F. Conine, *President*, High School, Montclair, N. J.

Miss Agnes Waring, *Vice-President*, High School, New Brunswick, N. J.

Andrew S. Hegeman, *Secretary-Treasurer*, Central High School, Newark, N. J.

The Suffolk County Mathematics Teachers Association was organized on November 10, 1937 at Sayville, Long Island. The officers are S. Manford Lloyd, Northport High School, President, Robert Folson, Huntington High School, Vice-President, and Harriet Burgie of Sayville High School, Secretary-Treasurer.

The third meeting of the Men's Mathematics Club of Chicago and the Metropolitan Area was held on December 17, 1937. Dr. O. E. Overn of Lake View High School spoke on "Revolutionary Changes in Algebra" and Mr. L. F. Summers on "Fire Insurance Schedule Ratings." Mr. Summers is Chief Engineer of America Fire Group of Insurance Companies.

The Program of the Detroit Mathematics Club for the year 1937–1938 is as follows:

December 2, 1937—Western High School

"Olney's Algebra," Dr. Paul Hickey, Detroit Institute of Technology, Detroit College of Law, Detroit, Michigan; Tea 3:30 P.M., address 4:30 P.M.

February 19, 1938—Noonday Luncheon Meeting

"Living Mathematics," Mary A. Potter, Supervisor of Mathematics, Racine, Wisconsin; Detroit Leland Hotel, 12:15 P.M.

March 31, 1938—Foch Intermediate School

"Unifying Themes in Mathematical Instruction," M. L. Hartung, Progressive Education, Ohio State University, Columbus, Ohio; Tea 3:30 P.M., address 4:30 P.M.

May 19, 1938—Northwestern High School

"Instruments Today and Yesterday," Eloise B. Voorheis, Mathematics Teacher, Libbey High School, Toledo, Ohio; Election of officers; Tea 3:30 P.M., address 4:30 P.M.

Officers

President, Hildegarde Beck
Vice President, Agnes Crow.
Secretary, Lorana Tremper.
Treasurer, Y. Basmadjian.
Chairman of Publicity, Deborah Jones.

The Association of Teachers of Mathematics of the Middle States met at Atlantic City on November 28, 1937.

President, Jonathan T. Rorer, William Penn High School, Philadelphia.

Secretary, Ruth Wyatt, Woodrow Wilson Junior High School, Philadelphia, Garden Room, Haddon Hall (Lounge Floor—Adjoining Vernon Room).

9:30 A.M.—Meeting of the Council.

10:00 A.M.—Meeting of the Association.

1. "Computing with Mathematical Inaccuracy" (45 minutes), Harry C. Barber, English High School, Boston, Mass.
2. "Practical Applications for Junior and Senior High Schools" (30 minutes), C. N. Shuster, State Teachers College, Trenton, N. J.
3. (a) "What a Progressive School is Doing to Prepare for the Alpha, Beta, Gamma Examinations." (b) "Suggested Modifications of Books VII, VIII, and IX of Solid Geometry" (15 minutes), Gordon R. Mirick, Lincoln School of Teachers College.
4. Discussion.